The Origins of QCD Confinement in Yang-Mills Gauge Theory

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1. What Makes Yang-Mills Gauge Theory Different from an Abelian Gauge Theory like QED?

In an Abelian Gauge Theory such as QED, a field strength two-form $F = \frac{1}{2!} F^{\mu\nu} dx_\mu \wedge dx_\nu = F^{\mu\nu} dx_\mu dx_\nu$ is expressed in terms of a potential one-form $A = A^\mu dx_\mu$ for a field of vector bosons, in this case photons, using the compact language of differential forms, as:

$$F = dA, \tag{1.1}$$

where $dA = \partial^\mu A^\nu dx_\mu \wedge dx_\nu = \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) dx_\mu dx_\nu \equiv \partial^{[\mu} A^{\nu]} dx_\mu dx_\nu$.

In Yang Mills theory, also known as non-Abelian gauge theory, there is an extra term in the field strength, and in particular, if the vector potential one-form is now $G = G^\mu dx_\mu$, then:

$$F = dG + igG^2, \tag{1.2}$$

where $G^2 = [G, G] = \frac{1}{2!}\left[G^\mu, G^\nu\right] dx_\mu \wedge dx_\nu = \left[G^\mu, G^\nu\right] dx_\mu dx_\nu$, and *g* is the group "running charge" strength.

The only difference is the existence of this extra term igG^2!

Mathematical Review Notes:

If you need a brief review of Yang Mills, remember that $F^{\mu\nu} \equiv T^i F_i^{\mu\nu}$ and $G^\mu \equiv T^i G_i^\mu$ are *N*x*N* matrices for any simple Yang-Mills group SU(N). The group structure is specified by $f^{ijk} T_i = -i\left[T^j, T^k\right]$ and the Latin internal symmetry index $i = 1,2,3 \ldots N^2 - 1$ is raised and lowered with the unit matrix δ_{ij}. Equation (1.2) can thus be expanded to the *N*x*N* equation $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig\left[G^\mu, G^\nu\right]$, and, with all components explicit, to the commonly-written $F^{i\,\mu\nu} = \partial^\mu G^{i\,\nu} - \partial^\nu G^{i\,\mu} - gf^{ijk} G_j^{\ \mu} G_k^{\ \nu}$. The T^i are often referred to as the group generators. In subsequent discussion, to diminish cluttering, we shall omit explicit rendering of the wedge products.

Regarding differential forms, recall also, that $dH = \frac{1}{p!} \partial_\nu H_{\mu_1 \mu_2 \ldots \mu_p} dx^\nu dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}$ defines the differential operator *d* as applied to any p-form *H*.

Finally, we will often use the commutator notation $[A, B] \equiv AB - BA$, and on occasion, the anticommutator $\{A, B\} \equiv AB + BA$.

2. What Do Source Current Source Densities Look Like in Yang-Mills Theory, versus QED?

In QED, one has both an electric and a magnetic current source (probability and flux) density. In forms language, the electric current source density is:

$$*J = d * F = d * dA, \tag{2.1}$$

which expands to the familiar $J^{\nu} = \partial_{\mu} F^{\mu\nu} = \partial_{\mu}\partial^{\mu} A^{\nu}$ with the gauge condition $\partial_{\mu} A^{\mu} = 0$, while the magnetic source current density is:

$$P = dF = ddA = 0, \tag{2.2}$$

the latter vanishing because $dd = 0$ for any two successive exterior derivatives. This expands to the familiar $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu} = 0$: there are *no magnetic charges* in QED.

In Yang-Mills theory, the source densities are related to the field strengths in the same manner, i.e., $*J = d * F$ and $P = dF$, but, because of the extra igG^2 term, we find in contrast, using (1.2), that:

$$*J = d * F = d * \left(dG + igG^2\right), \tag{2.3}$$

which expands to $J^{\nu} = \partial_{\mu} F^{\mu\nu} = \partial_{\mu}\partial^{\mu} G^{\nu} - \partial_{\mu}\partial^{\nu} G^{\mu} + ig\partial_{\mu}\left[G^{\mu}, G^{\nu}\right] = \partial_{\mu}\partial^{\mu} G^{\nu} + ig\left[G^{\mu}, \partial_{\mu} G^{\nu}\right] \neq 0$ using the gauge condition $\partial_{\mu} G^{\mu} = 0$. We also find that:

$$P = dF = d\left(dG + igG^2\right) = id\left(gG^2\right) \neq 0, \tag{2.4}$$

expanding to $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu} = i\left(\partial^{\sigma}\left(g\left[G^{\mu}, G^{\nu}\right]\right) + \partial^{\mu}\left(g\left[G^{\nu}, G^{\sigma}\right]\right) + \partial^{\nu}\left(g\left[G^{\sigma}, G^{\mu}\right]\right)\right)$.

While part of the magnetic source density still vanishes in the usual way because $ddG = 0$, *there is also a non-vanishing term in the magnetic source density*: $id\left(gG^2\right)$. Put differently: in Yang-Mills theory, magnetic sources do not vanish, as has been pointed out in the past by T'hooft & Polyakov and others. The "Yang-Mills electric" current density three-form $*J$ in (2.3) also acquires an extra term $idg * G^2$.

Might these non-vanishing magnetic three-forms (2.4) represent anything observed in the physical world?

Mathematical Review Notes:

Recall that $*J = *J^{\sigma\mu\nu} dx_{\sigma} dx_{\mu} dx_{\nu}$ and $*F = *F^{\mu\nu} dx_{\mu} dx_{\nu}$, which makes use of the duality formalism $*J^{\sigma\mu\nu} = \varepsilon^{\sigma\mu\nu\tau} J_{\tau}$ and $*F^{\mu\nu} = \frac{1}{2!}\varepsilon^{\sigma\mu\nu\tau} F_{\nu\tau}$ first developed by Reinich and Wheeler and later applied to differential forms by Hodge. Also, note that in Yang-Mills theory, $J^{\mu} \equiv T^{i} J_{i}^{\ \mu}$, $P^{\sigma\mu\nu} \equiv T^{i} P_{i}^{\sigma\mu\nu}$.

3. Boundary Integration Properties of Yang-Mills Magnetic Sources

Differential forms are tailor-made for examining surface and volume integrals over a closed boundary. So, to try to understand the magnetic three-form P of (2.4), we first examine the volume and surface integrals over P.

Taking the 3-volume integral of the P in (2.4), using $dd = 0$, and applying Gauss' law, enables us to rewrite (2.4) in integral form:

$$\oiiint P = \oiiint dF = \oiiint ddG + i\oiiint d\left(gG^2\right) = i\oiiint d\left(gG^2\right) = \oiint F = \oiint dG + i\oiint gG^2 \neq 0 . \qquad (3.1)$$

In part, the above employs Gauss' law, in the form $\oiiint d\left(gG^2\right) = \oiint gG^2$. In further part, the above contains the expression $i\oiiint d\left(gG^2\right) = \oiint dG + i\oiint gG^2$. Combining these two parts of (3.1), enables us to deduce that:

$$\oiint dG = 0 . \qquad (3.2)$$

Now, setting (3.2) into (3.1) yields a simplified version of (3.1):

$$\oiiint P = \oiint F = i\oiint gG^2 \neq 0 . \qquad (3.3)$$

Further, using (1.2) in (3.2), in the form $dG = F - igG^2$, and again using Gauss' law, now in the form $\oiint dG = \oint G$, yields an expanded version of (3.2):

$$\oiint dG = \oiint \left(F - igG^2\right) = \oiint F - i\oiint gG^2 = \oint G = 0 . \qquad (3.4)$$

Equations (3.3) and (3.4) tell us, mathematically, how these Yang-Mills magnetic sources behave at their boundaries. These two equations will be a primary focus of the discussion to follow.

How do we physically interpret (3.3) and (3.4)?

Mathematical Review Notes

Recall that Gauss' law for a given p-form H states that $\oint_d dH = \oint_{d-1} H$, where d is the dimensionality of the closed surface over which the integration takes place.

4. <u>An Important Gauge Symmetry over Closed Surfaces of Yang-Mills Magnetic Sources</u>

Although perhaps not immediately apparent, equation (3.2), $\oiint dG = 0$, tells us that there is no net flux of non-Abelian vector fields G^μ across any closed surface over the magnetic three-form source density *P*. To see this, subject the field strength two-form *F* to the transformation:

$$F \rightarrow F' = F - dG\,, \tag{4.1}$$

which expands to $F^{\mu\nu} \rightarrow F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\mu}G^{\nu]}$. Now we ask: what effect does the transformation (4.1) have *over a closed 2-dimensional surface* surrounding the magnetic three-form *P*, as well as *on the magnetic charge within the enclosed 3-dimensional volume*?

Substituting (4.1) into (3.3), we obtain:

$$\oiiint P = \oiint F \rightarrow \oiiint P' = \oiint F' = \oiint (F - dG) = \oiint F - \oiint dG = \oiint F = \oiiint P\,. \tag{4.2}$$

That is, under the transformation $F \rightarrow F' = F - dG$, we find that $\oiint F \rightarrow \oiint F' = \oiint F$ and $\oiiint P \rightarrow \oiiint P' = \oiiint P$.

The above reveals a very important, apparently unknown, gauge symmetry of Yang-Mills field theory. Consider, by way of contrast, that QED and related theories are invariant under the transformation $A^\mu \rightarrow A^\mu{}' = A^\mu + \partial^\mu \Lambda$. This means that the scalar "phase" Λ is not observable. Consider also by way of contrast, the that the field equations of gravitation are invariant under the (similar to (4.1)) gauge transformation $g^{\mu\nu} \rightarrow g^{\mu\nu}{}' = g^{\mu\nu} + \partial^{\{\mu}\Lambda^{\nu\}}$. This means that Λ^ν is not a gravitational observable.

So, when $\oiint F \rightarrow \oiint F' = \oiint F$ under the transformation $F^{\mu\nu} \rightarrow F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\mu}G^{\nu]}$, this means that the non-Abelian vector fields G^μ are not observable over any closed 2-D surface defined around the magnetic three-form *P*. More to the point: these is *no net flux of non-Abelian vector fields* G^μ across any closed surface containing *P*. In addition, $\oiiint P \rightarrow \oiiint P' = \oiiint P$ tells us that under the same transformation $F^{\mu\nu} \rightarrow F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\mu}G^{\nu]}$, the total magnetic charge within the specified 3-volume also does not change. More to the point: *this transformations does not remove any net magnetic charge out of the specified 3-volume.* All of these consequences emerge from $\oiint dG = 0$.

Finally, let's return to (3.3), which we expand to the form (see following (1.2)):

$$\oiiint P = \oiint F = i\oiint gG^2 = i\oiint g\left[G^\mu, G^\nu\right]dx_\mu dx_\nu = i\oiint g\left[G^\mu G^\nu - G^\nu G^\mu\right]dx_\mu dx_\nu \neq 0\,. \tag{4.3}$$

This is *non-zero*, which means that there *is* a net flux across the 2-D surface in the above, of whatever physical entities are represented by igG^2!

How might all of this relate to QCD confinement?

5. Possible Parallels with Four Main Features of QCD Confinement

There are four main features of QCD confinement, which appear to parallel the development of the previous section. These parallels are best specified with reference to baryons, as follows: Establish any **closed** surface over a baryon source density *P*. Then:

1) While gluons may flow ***within*** the closed surface across various ***open*** surfaces, *there can be no **net** flux of gluons in to or out of any **closed** surface.*

This may possibly be represented by $\oiint dG = 0$, and the invariance of $\oiint F \to \oiint F' = \oiint F$ under the transformation $F \to F' = F - dG$.

2) While quarks may flow ***within*** the closed surface across various ***open*** surfaces, *there can be no **net** flux of **individual** quarks in to or out of any **closed** surface.*

This may possibly be represented by the invariance of $\oiiint P \to \oiiint P' = \oiiint P$ under the transformation $F \to F' = F - dG$.

3) While there can be no ***net*** flux of individual quarks in to or out of any closed surface, there can indeed be a net flux of ***quark-antiquark pairs*** in to or out of any closed surface. The antiquark cancels the quark, thereby averting a ***net*** flux, and in this way, quarks do flow in to or out of the closed surface, ***but only paired with antiquarks, as mesons***.

This may possibly be represented as $i\oiint gG^2 \neq 0$.

4) It does not matter how hard or in what manner one "smashes" a baryon, one can still never extract a net flux of quarks or a net flux of gluons, but only a large number of meson jets.

This may be possibly represented by the fact that in all of the foregoing, the volume and surface integrals apply to *any and all closed surfaces*. One can choose a small closed surface, a large closed surface, a spherical closed surface, an oblong closed surface, and indeed, a closed surface of *any shape and size*. *The choice of closed surface does not matter.* These mathematical rules for what does and does not flow across any closed surface, in fact, thereby impose very stringent dynamical constraints on the behaviors of these non-Abelian magnetic sources: No matter what flows across various *open* surfaces, they may never be a *net* flux of anything across *any closed* surface. The only exceptions, which may flow across a closed surface, are physical entities represented by igG^2.

Where is the author going with this?

6. <u>What the Author Believes can be Proven to be True</u>

1. The magnetic three-form *P*, and its associated third-rank antisymmetric tensor $P^{\sigma\mu\nu}$, has all the characteristics of a baryon current density.

2. These $P^{\sigma\mu\nu}$, among their other properties, are naturally occurring sources *containing exactly three fermions*. These constituent fermions are most-sensibly interpreted as quarks.

3. $\oiint dG = 0$, or the surface symmetry $\oiint F \to \oiint F' = \oiint F$ under the transformation $F \to F' = F - dG$, tells us that there is no *net* flow of gluons across any *closed* surface over the baryon density.

4. The volume symmetry $\oiiint P \to \oiiint P' = \oiiint P$ under $F \to F' = F - dG$, tells us that there is no *net* flow of quarks across any *closed* surface over the baryon density.

5. The physical entities represented by igG^2, when examined in further detail, have the characteristics of mesons.

6. $i\oiint gG^2 \neq 0$ tells us that mesons are the only entities which may flow across any *closed* surface of the baryon density.

But, there is one remaining question of paramount importance:

What about the "Mass Gap"?

7. <u>How do we Fill the Mass Gap?</u>

The one question remaining is how the mesons may become *massive*, given the seemingly-massless nature of the gluons. This is necessary to explain why the nuclear force is strong but short-ranged, thereby filling the so-called "mass gap."

This problem may be solved starting with equation (2.3), $*J = d*F = d*\left(dG + igG^2\right)$, which we have not yet explored in depth here. Expanded, and with $\partial_\mu G^\mu = 0$, (2.3) is:

$$J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu G^\nu - \partial_\mu \partial^\nu G^\mu + ig\partial_\mu \left[G^\mu, G^\nu\right] = \partial_\mu \partial^\mu G^\nu + ig\left[G^\mu, \partial_\mu G^\nu\right], \tag{7.1}$$

which we abbreviate, functionally, as $J^\nu = F\left(G^\nu\right)$.

The trick to solving the mass gap, is to make use of the above-noted symmetries under $F \rightarrow F' = F - dG$, and also, to exactly obtain the *inverse* relationship $G^\nu = F^{-1}\left(J^\nu\right)$. In QED, where the term igG^2 does not exist, and A^ν, J^ν, and p^μ are all simple four-vectors, this is trivial, because in summary, one starts with $J^\nu = \partial_\mu \partial^\mu A^\nu$, uses $\partial_\mu \rightarrow iq_\mu$ to turn this into $J^\nu = -q_\mu q^\mu A^\nu$, and then "inverts," to obtain $A^\nu = -\frac{1}{q_\mu q^\mu} J^\nu$, which is also connected in a known way to the photon propagator $-\frac{ig_{\mu\nu}}{q_\mu q^\mu}$. The term $q_\mu q^\mu$ is easily put into the denominator, because $q_\mu q^\mu$ is a scalar.

Non-Abelian gauge theory is trickier, because a) there is the extra term igG^2, b) the $G^\mu \equiv T^i G_i^{\ \mu}$ and $J^\mu \equiv T^i J_i^{\ \mu}$ are *matrices* of four-vectors, and not the simple non-matrix four-vectors A^ν and J^ν of QED, c) the four-momentum vectors $p^\mu \equiv T^i p_i^{\ \mu}$ which are analogs of q^μ, are also *matrices* of four-vectors, and d) because of these matrices, one must be very careful to employ commutators when performing the analog to the $\partial_\mu \rightarrow ip_\mu$ substitution.

But most importantly, the aforementioned matrix character of A^ν, J^ν, and p^μ means that the $J^\nu = -q_\mu q^\mu A^\nu$ of QED will migrate over to the form $J^\nu = (\mathrm{N \times N\ Matrix})G^\nu$ for SU(N) in general, and that one must then obtain the *matrix inverse* of this $\mathrm{N \times N\ Matrix}$ to obtain $G^\nu = F^{-1}\left(J^\nu\right)$. For the special case of SU(2), this is a diagonal matrix with each diagonal element identical, so inversion is simple. This is why it has proven possible to do accurate calculations of vector boson masses in weak and electroweak theory. But for SU(3) and larger, this matrix is non-diagonal and non-trivial. What one normally thinks of as the propagator, is now an $\mathrm{N \times N\ Matrix}$, specifically related to the inverse matrix $(\mathrm{N \times N\ Matrix})^{-1}$. So, when one finally gets to $G^\nu = F^{-1}\left(J^\nu\right)$, one has an equation of the form $G^\nu = (\mathrm{N \times N\ Matrix})^{-1} J^\nu$. When used in amplitudes $\mathfrak{M} \sim J^\mu g_{\mu\nu} (\mathrm{N \times N\ Matrix})^{-1} J^\nu$, together with suitable SU(N) scalar multiplets which break symmetry similarly to how this is done in electroweak theory, the result is that the mesons become massive, they also obtain imaginary mass components which give them a short lifetime, the interactions become short range, and the mass gap can be filled.

What About Some Pictures?

8. Some Feynman-Type Diagrams that Result from all of this

While we will not show the detailed development here, the above can be developed into the following three Feynman-type diagrams:

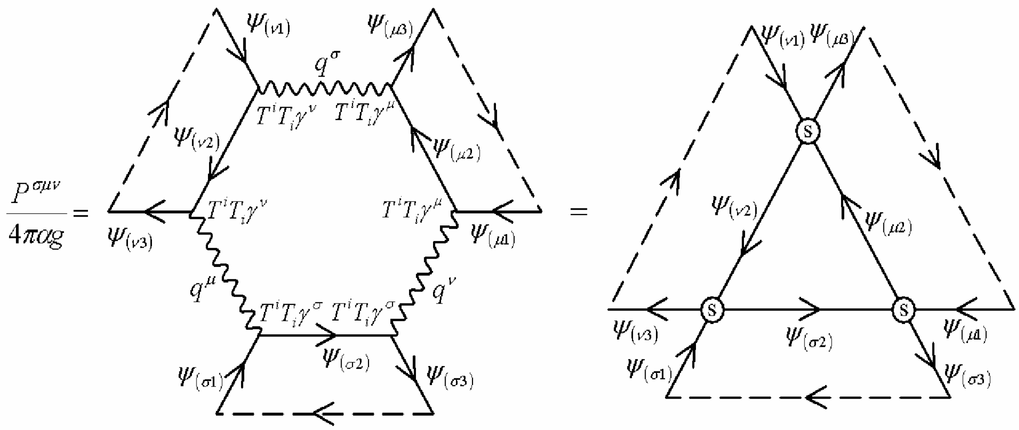

Figure 1: The Yang-Mills Magentic Source Density as a Three-Fermion Baryon Density

$$\frac{q_{(\mu)}^{2} q_{(\nu)}^{2}}{4\pi\alpha_S}G^2 = \frac{q_{(\mu)}^{2} q_{(\nu)}^{2}}{4\pi\alpha_S}\left[G^\mu, G^\nu\right] = \frac{1}{4\pi\alpha_S}\left[J^\mu, J^\nu\right] = \left[\bar{\psi}_{(\mu3)}T^iT_i\gamma^\mu\psi_{(\mu2)}, \bar{\psi}_{(\nu2)}T^iT_i\gamma^\nu\psi_{(\nu1)}\right]$$

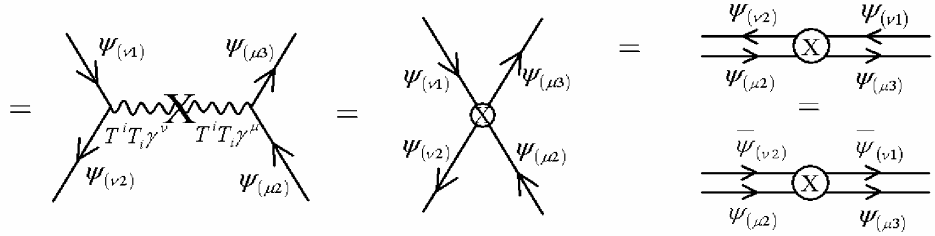

Figure 2: The Origin of Unstable Mesons in the Yang-Mills G^2 Term

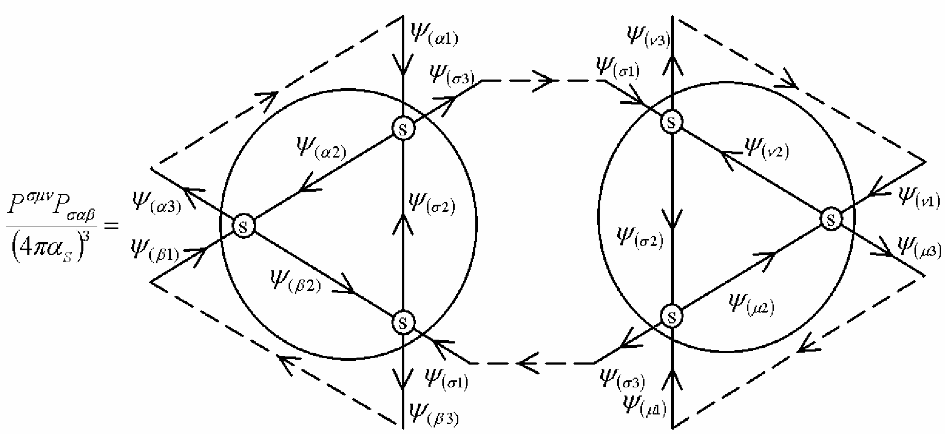

Figure 3: Interaction Between two Baryons via Meson Exchange

Thank you for listening!