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Study of connections between neutrino phenomenology and leptogenesis in an SO(10) inspired context

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Introduction

One of the greatest shortcomings of standard cosmological models is the absence of a definite explanation of the asymmetry between matter and antimatter in the Universe. It is possible to trace back the relative proportions of the nuclei to a single parameter, which can be interpreted as the difference between the number density of baryons and antibaryons; such parameter, however, remains unexplained from first principles in standard cosmology.

The question which is posed is therefore whether it is possible to start with a net zero baryonic number and end up with a non vanishing one. It is clear that such a result requires some specific conditions on the nature of the processes happening in nature; in particular, they have been outlined in detail by Sakharov [1]. Although at a qualitative level these conditions are already present in the Standard Model, it has been found that quantitatively they are not able to produce the correct baryon asymmetry which is experimentally measured.

It turns out, as will be discussed in detail in this work, that a possibility for the realization of these conditions is to simply extend the Standard Model with the addition of massive particles subject to CP-violating decays into states with a non vanishing lepton number; this accounts for the generation of a net leptonic number, which is then converted, through a process of non perturbative nature already present in the Standard Model, into a net baryonic number. The whole scenario is known as *baryogenesis via leptogenesis* [2]; a particularly attractive possibility lies in the identification of the massive particles with the right-handed neutrinos in the context of the seesaw mechanism proposed by Minkowski [3] and Gell-Mann, Ramond and Slansky [4].

Since right-handed neutrinos appear naturally in the grand unified model based on the group $SO(10)$ [5], it is of interest to discuss leptogenesis under the constraints suggested by such a model. It turns out, however, that such constraints render a successful leptogenesis extremely difficult to obtain. This happens because, unless a fine tuning on the neutrino mass parameters is introduced, the right-handed neutrinos become very hierarchical in mass, with the lowest mass being too small to allow for leptogenesis. A compactness in the right-handed neutrino mass spectrum is, however, able to overcome this difficulty and achieve a consistent leptogenesis.

Our task in this work may thus be described as follows; in the context of a specific seesaw model, whose features are inspired by the mass relations naturally arising in the context of $SO(10)$, we have derived the relations between the right-handed neutrino mass spectrum and the parameters of the Standard Model neutrino masses and mixing. We briefly recall (this will be described in detail below) that such parameters consist of the masses of the three neutrinos, three mixing angles, a CP violating phase and two Majorana phases; only the squared mass differences can be fixed from experiments on mixing, so the lightest of the masses remains unknown (although of course bounds on it are known mainly from beta decay experiments). Random values for the mixing angles have been generated inside the experimentally allowed ranges, and for each generation a compactness condition has been imposed on the right-handed neutrino mass spectrum to determine the other parameters. We have looked for correlations between the parameters which allow such a compact spectrum.

On these points in the space of parameters a further constraint has been imposed for successful leptogenesis; through the kinetic equations the numerical values for the baryon number density have been deduced in order to compare them to the experimentally known values and exclude then some of these points.

The main purpose of this work has therefore been that of restricting the range of neutrino mass parameters through constraints coming from the assumption that they are at the same time responsible for the genesis of the asymmetry between matter and antimatter.

Chapter 1

Standard Model of Particles and some extensions

1.1 The Standard Model

The Standard Model of particles [6] is based on the simple assumption that particles are classified into representations of the local gauge symmetry group $SU(3) \times SU(2) \times U(1)$; for this presentation, we will simply state the various representations found in nature. The behavior of a given field under $SU(3)$ and $SU(2)$ is specified simply by the dimension of the representation to which it belongs; we recall that, given a multiplet ψ, a gauge transformation acts on it as:

$$\psi' = e^{igt_a\theta^a(x)}\psi \tag{1.1}$$

where g is the coupling constant, t_a are the group generators and $\theta^a(x)$ are the gauge parameters. Given the free field Lagrangian, a gauge invariant Lagrangian can then be obtained by the so called minimal coupling through the substitution of the ordinary derivative ∂_μ by the covariant derivative $\partial_\mu - igA^a_\mu t_a$. Here A^a_μ is a vector field belonging to the adjoint representation of the gauge group which transforms under (1.1) as:

$$A'^a_\mu(x)t_a = e^{igt_b\theta^b(x)}A^a_\mu(x)t_a e^{-igt_b\theta^b(x)} + \partial_\mu\theta^a(x)t_a \tag{1.2}$$

In the case of $U(1)$, the gauge transformation takes the form:

$$\psi' = e^{iYg'\theta(x)}\psi \tag{1.3}$$

where g' is a universal coupling constant and Y is the hypercharge, which is different for each particle.

Strong interactions are described in the Standard Model by coupling with the eight gauge fields of $SU(3)$; weak and electromagnetic interactions are obtained as a result of a mixing of the coupling with the three gauge fields of $SU(2)$ and the single gauge field of $U(1)$. This mixing produces a coupling with the electromagnetic field proportional to the electromagnetic charge, which is linked to the generators of $SU(2) \times U(1)$ through the famous Gell-Mann-Nishijima relation[1]:

$$Q = \frac{Y}{2} + \frac{t_3}{2} \tag{1.4}$$

where Q is the charge and t_3 is the eigenvalue of the third generator of $SU(2)$ (we take the convention $t_a = \sigma_a$).

The complete Lagrangian of the Standard Model can thus be written as:

$$\mathcal{L} = \mathcal{L}_{matter} + \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \tag{1.5}$$

The kinetic terms will not be written hereafter.

Matter fields

Because of the parity violation of weak interactions, left-handed and right-handed fields generally behave differently under the $SU(2) \times U(1)$ part of the gauge group. Leptons, which do not interact strongly, are singlets of $SU(3)$; left-handed leptons (electron and electron neutrino), which interact weakly, transform as a **2** under $SU(2)$ and have hypercharge -1. The Standard Model considers neutrinos to be massless; under this assumption, it is well known that right and left-handed components of the field decouple. In the Standard Model, the right-handed component of neutrinos is considered to be missing; therefore, we only need to add the massive leptons right-handed fields, which are singlets under $SU(2)$ and have $Y = -1$.

Quarks, which do interact strongly, come under the representation **3** of $SU(3)$; their left-handed components are organized into doublets of $SU(2)$ with hypercharge $Y = 1/3$. The right-handed components are singlets under $SU(2)$ and have hypercharge $4/3$ and $-2/3$. As is well known, there are three families of quarks and three families of leptons; therefore, all matter particles in the Standard Model are organized into the structures in Table 1.1.

Each of these representations is repeated three times because of the presence of three different families ($\begin{pmatrix} u \\ d \end{pmatrix}$, $\begin{pmatrix} c \\ s \end{pmatrix}$, $\begin{pmatrix} t \\ b \end{pmatrix}$ for quarks and $\begin{pmatrix} e \\ \nu_e \end{pmatrix}$, $\begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}$, $\begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$ for leptons).

Higgs and $SU(2) \times U(1)$ gauge

A further doublet of complex scalar fields is needed in order to give mass to three of the gauge fields of $SU(2) \times U(1)$ through the Brout-Englert-Higgs mechanism

[1] The Gell-Mann-Nishijima relation of course refers to the identical relation derived for the $SU(2)$ group of isospin; they however express exactly the same physical content.

Particles	$SU(3)$	$SU(2)$	Y
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{3}$
u_R	$\mathbf{3}$	$\mathbf{1}$	$\frac{4}{3}$
d_R	$\mathbf{3}$	$\mathbf{1}$	$-\frac{2}{3}$
$l_L = \begin{pmatrix} e_L \\ \nu_{eL} \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	-1
e_R	$\mathbf{1}$	$\mathbf{1}$	-2

Table 1.1: Matter content of the Standard Model

[7, 8]. This Higgs field ϕ does not couple to $SU(3)$ gauge fields; moreover, we want it to be a neutral particle[2]. Without loss of generality we can choose a gauge in which the Higgs field has only the $t_3 = -1$ component; then (1.4) shows that, in order for the field to be electrically neutral, it should have hypercharge $+1$. In order for the field to spontaneously break $SU(2)$ symmetry, we also introduce a potential which is minimized at $|\phi|^2$ equal to a certain vacuum expectation value. The previous symmetry considerations constrain the part of the Lagrangian involving $SU(2) \times U(1)$ gauge fields and the Higgs field to the following form[3]:

$$\mathcal{L}_{MS} = \left| \partial_\mu \phi - igW^a_\mu \frac{t_a}{2}\phi - i\frac{g'}{2}B_\mu \phi \right|^2 + \mu^2|\phi|^2 - \lambda^2|\phi|^4 \tag{1.6}$$

(the factor of 2 in the definition of the generator of $SU(2)$ is conventional).

The form of the potential for ϕ is chosen so that symmetry is spontaneously broken in the ground state: to see this at a quantum level would require to explicitly compute the effective potential [9]. At the classical level, however, it is easily seen that the potential is minimized for $|\phi| = v = \frac{\mu}{\sqrt{2}\lambda}$, and we are allowed to choose the gauge in such a way that $\phi = \begin{pmatrix} 0 \\ v + \eta \end{pmatrix}$. After the substitution, separating out the terms which do not contain η, we obtain:

$$\frac{g^2}{4}v^2 \left(W^{+\mu}W^-_\mu + (W^3)^2 \right) + \frac{g'^2}{4}B^2 v^2 - \frac{gg'}{2}v^2 W^3_\mu B^\mu \tag{1.7}$$

where $W^\pm = W^1 \pm iW^2$. This is easily rewritten as:

$$\frac{g^2}{4}v^2 W^+ W^- + \frac{1}{4}v^2 \left(gW^3 - g'B \right)^2 \tag{1.8}$$

[2]This requirement comes from the physical observation that the photon is massless.

[3]The form of the potential is the simplest one, and is the only one which does not involve non renormalizable terms.

We have obtained the result that three of the generators of $SU(2) \times U(1)$ are massive, while the other, namely $g'W^3 + gB$, has remained massless. This last one is of course to be identified with the photon field; after using the correct normalization, we find the electromagnetic field:

$$A = \frac{gB + g'W^3}{\sqrt{g^2 + g'^2}} \tag{1.9}$$

The other three generators are instead identified with the three massive vector bosons of the weak interaction. Since (1.9) has the form of a rotation, it is usually parameterized in terms of an angle of rotation known as the Weinberg angle:

$$\tan\theta_w = \frac{g'}{g} \tag{1.10}$$

Yukawa couplings

We can use the same Higgs field to give mass to the other matter fields of the Model. Mass terms couple left-handed with right-handed fields; this means that we have to introduce Yukawa couplings between the Higgs field, the left-handed doublets and the right-handed fields. For the leptons, due to the existence of a single right-handed field per family (e_R in the first family), there is a single possible term, which is written as $(\bar{l}_L\phi)e_R$ (l_L is the left-handed lepton doublet). For the quarks, due to the existence of two right-handed fields, we have two possible terms; in order for these to give the correct mass terms, we need to introduce the conjugate field to the Higgs $\phi^c = \begin{pmatrix} v+\eta \\ 0 \end{pmatrix}$. Experimentally, it is found out that mass eigenstates are generally linear combination of the three generations weak interaction eigenstates; therefore we now explicitly introduce a family index i which runs from 1 to 3. The part of the Lagrangian which, upon spontaneous symmetry breaking, will give mass to the matter fields is:

$$-Y_l\left(\bar{l}_L\phi\right)e_R - Y_u^{ij}\left(\bar{q}_{iL}\phi^c\right)u_{jR} - Y_d^{ij}\left(\bar{q}_{iL}\phi\right)d_{jR} + h.c. \tag{1.11}$$

where q_{iL} are the three left-handed quark doublets, $u_{jR} = (u_R, c_R, t_R)$ and $d_{jR} = (d_R, s_R, b_R)$.

After substituting for ϕ its vacuum expectation value, the Lagrangian will contain mixing terms between the weak eigenstates of quarks; mass eigenstates are determined by diagonalizing this part of the Lagrangian. The two matrices Y_u^{ij} and Y_d^{ij} are not necessarily symmetric; they can be put into a diagonal form through the transformation $Y_{u,d} = U_{u,d}M_{u,d}K^\dagger_{u,d}$, where M are diagonal matrices and U and K are unitary matrices. The matrices $U_{u,d}$ are then used as matrices of a basis change from the weak eigenstates to the mass eigenstates; upon reexpressing the weak interaction part of the Lagrangian in the mass basis, we find that the interaction produces a mixing between mass eigenstates through the so called Cabibbo-Kobayashi-Maskawa (CKM) matrix [10, 11]:

$$V = U_u^\dagger U_d \tag{1.12}$$

Through the CKM matrix, the weak interaction coupling terms involving quarks can be written in terms of the mass eigenstates (which, by an abuse of notation, we will once again call u and d) as:

$$g\left(W_\mu^+ \overline{u}_L^i \gamma^\mu V_{ij} d_L^j + h.c.\right) \tag{1.13}$$

The CKM is a complex 3-by-3 unitary matrix, so it has 9 real parameters. We can redefine the six left-handed fields by a $U(1)$ transformation; if we redefine all of them, however, the matrix V is left unchanged. After redefining five of them, we are left with 4 parameters, which are conveniently chosen according to the following convention:

$$\begin{aligned} V = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \\ &\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{1.14}$$

1.2 Neutrino masses and mixing

In the Standard Model neutrinos appear only as left-handed fields; as is well known, this requires necessarily that neutrinos are massless, since a Dirac mass term couples left-handed and right-handed fields.

Neutrino oscillations

As evidenced for the first time by the Super-Kamiokande experiment [12], neutrinos are subject to oscillations between different flavours; by this we mean that a neutrino created with a specific lepton flavor is later measured to have a different flavor. This implies that the weak eigenstates ν_e, ν_μ, ν_τ which we have used so far are not eigenstates of the Hamiltonian for free neutrinos, and therefore are not mass eigenstates. Let us exemplify how the presence of a mixing mass term could produce flavor oscillations by a simple two state example; suppose that the mass part of the Lagrangian is diagonalized by the two states:

$$|\nu_1\rangle = \alpha\,|\nu_e\rangle + \beta\,|\nu_\mu\rangle\,, |\nu_2\rangle = \alpha^*\,|\nu_\mu\rangle - \beta^*\,|\nu_e\rangle \tag{1.15}$$

with masses respectively m_1 and m_2. Then, if a neutrino is produced in the weak eigenstate $|\nu_e\rangle$, its subsequent time evolution will be given, after decomposition on the mass eigenstates:

$$|\psi(t)\rangle = \alpha^*\,|\nu_1\rangle\, e^{-iE_1 t} - \beta\,|\nu_2\rangle\, e^{-iE_2 t} \tag{1.16}$$

where $E_i = \sqrt{p^2 + m_i^2}$. Multiplying on the left by $\langle \nu_e |$ and $\langle \nu_\mu |$ and taking the square moduli, it is seen that the probabilities of finding the neutrino in one of the two weak eigenstates oscillates periodically with a frequency $|E_1 - E_2|$; since neutrino masses are expected to be extremely small, we can expand the square root in the dispersion relation to obtain:

$$|E_1 - E_2| \approx \frac{\left|m_1^2 - m_2^2\right|}{2p} \tag{1.17}$$

The results of this simple model are straightforwardly generalizable to the complete three-flavor case; it is then evident that oscillation experiments allow us to have information on the absolute squared mass differences between different species[4]. The matrix relating weak eigenstates to mass eigenstates is known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [13]. The parametrization generally used for the PMNS matrix is very similar to the one introduced in Section 2.1 for the CKM matrix; the only difference is that, to take into account the possibility that neutrinos are described by Majorana spinors, there are two additional phases (apart from a global phase multiplying the whole matrix) which cannot be removed by a redefinition of the fields. We will often need the explicit form of this parametrization:

$$\begin{aligned} U = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \\ &\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \end{aligned} \tag{1.18}$$

where, of course, the angles are different from and independent of the corresponding angles in the CKM matrix.

If we make the additional assumption that for the heavy leptons e, μ and τ weak eigenstates coincide with mass eigenstates, then the interaction terms become, in terms of the mass eigenstates ν^i:

$$g\left(W_\mu^+ \bar{e}_L^i \gamma^\mu U_{ij} \nu^j + h.c.\right) \tag{1.19}$$

The problem of introducing a mass term for neutrinos is far from trivial; it is clear that a Dirac term would require the introduction of a right-handed field. Then we could add to the Lagrangian a term of the same form as for the quark sector, thus giving a Dirac mass to neutrinos. The disadvantage of this possibility is that it does not explain the smallness of neutrino masses; it would be more aesthetically appealing to find a mechanism which leads naturally to small values for these masses.

[4] We mention here that in literature, because of the way they were historically measured, the two squared mass differences δm^2 and Δm^2 and the angles θ_{12} and θ_{23} are referred to as solar and atmospheric parameters.

A Majorana mass could be given to neutrinos without the right-handed field; in order to do that without breaking $SU(2)$ invariance, however, the minimal choice is the non renormalizable Weinberg operator [14]:

$$M_{ij}\left(\bar{l}_L^i \phi^c\right)\left(\phi^c l_L^i\right)^\dagger \tag{1.20}$$

Diagrammatically this is represented in Figure 1.1.

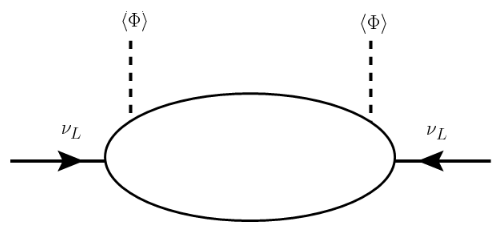


Figure 1.1: Diagrammatical representation of the Weinberg operator; the ellipse stands for the renormalizable mechanism expected to generate the effective vertex.

After spontaneous symmetry breaking, this term translates into a Majorana mass term for neutrinos. In order for neutrinos to have masses consistent with experimental bounds, the coefficients M_{ij} should of course be very small. The non renormalizability of the Weinberg operator and the smallness of the neutrino mass can both be explained by the presence of a renormalizable interaction characterized by a large mass scale M; then neutrino masses will be suppressed by a factor M^{-1}, and the low-energy physics will be characterized by the Weinberg operator.

An important feature of (1.20) is that, in the hypothesis that neutrinos are described by Dirac spinors with a Majorana mass term, lepton number is not conserved. If neutrinos were instead described by Majorana spinors with a Majorana mass term, then they are forced to carry zero lepton number, so the term in (1.20) conserves lepton number; anyway, in weak interactions lepton number would not be conserved[5]. Thus the only way for lepton number to be conserved would be if neutrinos were described by Dirac spinors with Dirac mass terms.

Seesaw mechanism

Our aim is to look for a renormalizable mechanism which is able to reproduce, in the low-energy limit, the Weinberg operator. A simple possibility is given by the so called seesaw mechanism [4]; this requires the introduction of a heavy right-handed neutrino which couples to the left-handed neutrino and to the Higgs boson through the following $SU(2)$ invariant terms (for simplicity we will restrict to a single flavor):

$$-\frac{1}{2}M\overline{N}^c_R N_R - \lambda\left(\bar{l}_L\phi^c\right)N_R + h.c. \tag{1.21}$$

The new coupling gives rise to the diagram in Figure (1.2); we can compute the amplitude of the diagram to obtain an effective Weinberg operator for neutrinos which is suppressed by a factor $\frac{1}{M}$ (the same result can also be obtained by explicit integration of the N_R degree of freedom). Let us derive the same result in another way.

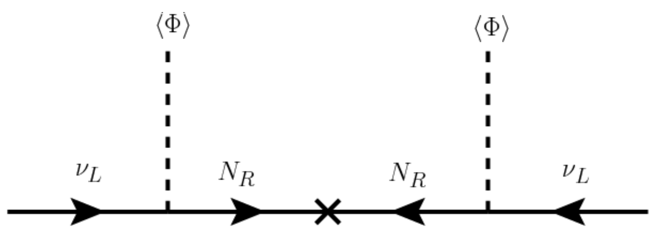


Figure 1.2: Diagrammatical representation of the Weinberg operator in the context of seesaw model.

After spontaneous symmetry breaking, the Higgs field acquires a vacuum expectation value, so that in (1.21) the following terms appear:

$$-\frac{1}{2}M\overline{N}^c_R N_R - \lambda v\overline{\nu}_L N_R + h.c. = -\frac{1}{2}\begin{pmatrix}\overline{N}^c_R & \overline{\nu}_L\end{pmatrix}\begin{pmatrix}M & \lambda^* v\\ \lambda v & 0\end{pmatrix}\begin{pmatrix}N_R\\ \nu_L\end{pmatrix} \tag{1.22}$$

The matrix in (1.22) can now be diagonalized to give the mass eigenstates; the eigenvalues of the matrix, which are the masses of the two eigenstates, turn out to be approximately M and $\frac{v^2|\lambda|^2}{M} = \frac{|M_D|^2}{M}$ (the notation M_D is chosen to underline that the coupling term between ν_L and N_R has the form of a Dirac

[5]This is evident, for example, in beta decay, where an electron and an antineutrino are produced; if the antineutrino brings zero lepton number, then this quantity is not conserved.

mass term). We are once again led to the conclusion that the mass of neutrinos, in the seesaw mechanism, is suppressed by a factor $\frac{1}{M}$.

In the seesaw mechanism, as we noted at the end of the above section, lepton number conservation is naturally violated; this will be an important feature in the context of baryogenesis.

We have not commented on the nature of the spinors involved in the seesaw mechanism, whether they are Dirac or Majorana spinors. Both possibilities are admitted by theory; for the following we will assume that they are Majorana spinors.

In the case of seesaw with different flavors, M and M_D are promoted to matrices with flavor indices, and the neutrino mass matrix becomes:

$$M_\nu = M_D^T M^{-1} M_D \tag{1.23}$$

In (1.23) we have made the hypothesis that there was a right-handed neutrino per family. It is clear that for each right-handed neutrino a left-handed neutrino becomes massive; since the Super Kamiokande experiment provides us with information only on squared mass differences, there is still the possibility that only two of the neutrinos are massive, and thus that there exist only two right-handed neutrinos. As we will see in the next section, however, in the context of Grand Unified Theories, it is natural to assume that there exists a right-handed neutrino per family.

The mechanism which we have described is known in literature as the seesaw type I, in order to distinguish it from similar possibilities; we mention here the seesaw type II [15] since it naturally appears in $SO(10)$ models. This mechanism requires the introduction of an $SU(2)$ triplet of scalar particles interacting both with leptons and with the Higgs through Yukawa couplings. The exchange of these particles generates an effective coupling between leptons and Higgs which has the form of the Weinberg operator, and is thus able to endow neutrinos with Majorana masses.

1.3 Grand Unified Theories

It is tempting to hypothesize that the gauge group of the Standard Model could be embedded into a larger simple group. The advantages of such a possibility are not merely aesthetic. In the Standard Model the $U(1)_Y$ part of the gauge group has generators which are not quantized; this implies that the quantization of the electric charge comes as a mere coincidence. We will see that, in the context of Grand Unified Theories, electric charge becomes a combination of generators of larger groups, and thus it becomes naturally quantized.

SU(5) Grand Unification

Since the standard $SU(3) \times SU(2) \times U(1)$ admits four diagonal generators (hypercharge Y, the third generator of $SU(2)$ t_3 and the two generators λ_3 and

λ_8 of $SU(3)$), it is clear that the larger gauge group must have at least rank 4; it should also contain as a subgroup the standard $SU(3) \times SU(2) \times U(1)$. Among the groups which are simple or can be written as products of simple groups, the only possibility is $SU(5)$ [16]; we follow the treatment given in [17].

The way in which representations in $SU(5)$ are decomposed into representations of the Standard Model is easily written down; in the fundamental representation ψ^α (where α runs from 1 to 5) we interpret the indices from 1 to 3 as color indices of $SU(3)$ and the indices 4 and 5 as isospin indices of $SU(2)$. Then it is evident that the $\mathbf{5}$ of $SU(5)$ decomposes as a $(\mathbf{3}, \mathbf{1})$ and a $(\mathbf{1}, \mathbf{2})$ (we have indicated the dimensions of the $SU(3)$ and $SU(2)$ representations). Since the $\mathbf{3}$ of $SU(3)$ must correspond to quarks, and we know that left-handed quarks come into doublets of $SU(2)$, we find that $\overline{\mathbf{5}}$ contains a triplet of $SU(3)$ which can be identified with a right-handed quark (the d quark for the lightest family) and a leptonic doublet (which is identified with ν_{eL} and e_L for the lightest family). By analogous reasoning, we find that the other matter particles of the Standard Model can be fit into the $\mathbf{10}$ representation, which decomposes into a $(\overline{\mathbf{3}}, \mathbf{1})$ (the right-handed u quark), a $(\mathbf{1}, \mathbf{1})$ (the right-handed electron) and a $(\mathbf{3}, \mathbf{2})$ (the two left-handed u and d quarks).

Since we have identified the 4 diagonal generators of the Standard Model with the 4 roots of $SU(5)$, we find that electric charge is naturally quantized and traceless. This implies the fundamental relation $3Q_d - Q_e = 0$.

Of the 24 generators of $SU(5)$, only 12 are identifiable with the generators of the Standard Model. The other 12 must evidently be made unobservable at low energies by a spontaneous symmetry breaking mechanism; this can be realized through the introduction of a $\mathbf{24}$ scalar field which, in an analogous way to the spontaneous breaking of $SU(2)$ symmetry, gives mass to the 12 fields which are not present in the Standard Model. Since these fields mix color and isospin indices, they are able to produce baryon number violating processes such as the proton decay. The fact that these processes have never been observed put a constraint on the masses of such bosons, which is the scale at which grand unification happens, to be greater than $10^{15}\ GeV$.

Mass terms for fermions in $SU(5)$ are produced by spontaneous symmetry breaking just as in the Standard Model; the minimal choice requires the introduction of a Higgs field belonging to the representation $\mathbf{5}$. If we denote the $\overline{\mathbf{5}}$ and the $\mathbf{10}$ matter fields by ψ_μ and $\phi^{\mu\nu}$ (the latter is antisymmetrical in its indices) and the Higgs field by χ^μ, then the mass term will originate from the Yukawa couplings:

$$\lambda_1 \overline{\psi}_\mu \phi^{\mu\nu} \chi^*_\nu + \lambda_2 \overline{\phi}^{\mu\nu} \phi^{\alpha\beta} \chi^\delta \epsilon_{\mu\nu\alpha\beta\delta} \tag{1.24}$$

By explicit substitution we can now find the mass relations $M_d = M_e$, $M_s = M_\mu$ and $M_b = M_\tau$.

These relations are to be verified at the unification mass scale of $10^{15}\ GeV$; by renormalization group arguments it is seen that the last one is well verified, while the others are violated. It is possible to adjust this prediction by introducing another $\mathbf{45}$ Higgs field which gives more freedom and allows to maintain the last prediction while modifying the others; Georgi and Jarlskog [18] have shown that the use of three **5** and a **45** leads to mass relations of the form $M_b = M_\tau$, $M_\mu = 3M_s$ and $M_e = \frac{M_d}{3}$, which are instead roughly verified at the unification scale.

Let us derive another fundamental consequence of $SU(5)$ grand unification. The covariant derivative in $SU(5)$ is characterized by a single coupling g_5; among the generators of the group 3 will correspond to the three generators of $SU(2)$ in the space of the two indices 4 and 5 and another one will correspond to hypercharge, with a coefficient properly chosen so that the generators satisfy the relation:

$$Tr\,\{t_a, t_b\} = 2\delta_{ab} \tag{1.25}$$

This implies that the fermionic coupling with the generators of the Standard Model $SU(2)$ and $U(1)$ will be completely specified in terms of the coupling g_5. It is then possible to obtain a prediction for the value of the Weinberg angle at the scale of Grand Unification; it turns out that $\tan\theta_w = \sqrt{\frac{3}{5}}$.

Pati-Salam Grand Unification

A completely different possibility of Grand Unification lies in the observation that the Standard Model explicitly breaks left-right symmetry, since the weak interaction involves the group $SU(2)_L$. It is possible to restore left-right symmetry by introducing another group $SU(2)_R$; in order to do that, we clearly have to introduce a right-handed neutrino which has to be the $SU(2)_R$ partner of the electron. The Pati-Salam model of unification [19] further generalizes the $SU(3)$ gauge group to a $SU(4)$, where the fourth "color" is identified with the leptonic doublet. The matter content of this unified model is now clearly a $(\mathbf{4},\mathbf{2},\mathbf{1})$ (where we have indicated the dimension of the representation under $SU(4)$, $SU(2)_L$ and $SU(2)_R$ respectively) for the left-handed quarks and leptons and a $(\mathbf{4},\mathbf{1},\mathbf{2})$ for the right-handed quarks and leptons.

As in the previous section, we need a Higgs field in order to break $SU(4)$ symmetry to $SU(3)\times U(1)$ and to break $SU(2)_R$; clearly, the minimal choice is $(\mathbf{4},\mathbf{1},\mathbf{2})$.

In order to give mass to fermions, another Higgs field needs to be introduced; since such a field has to couple the $\overline{(\mathbf{4},\mathbf{2},\mathbf{1})}$ and the $(\mathbf{4},\mathbf{1},\mathbf{2})$, the minimal choice is a $(\mathbf{1},\mathbf{2},\mathbf{2})$.

$SO(10)$ Grand Unification

Both $SU(5)$ and the Pati-Salam model can be embedded into a larger, simple group, $SO(10)$ [5]. This embedding is slightly more complicated than the previous, due to the fact that $SO(10)$ is an orthogonal group. As a matter of fact, many alternative routes can be found which break $SO(10)$ to the Standard Model gauge group; these are summarized in Figure 1.3. To fix ideas, we will concentrate on the breaking of $SO(10)$ to $SU(5)$. We follow here the treatment given in [20] and [21].

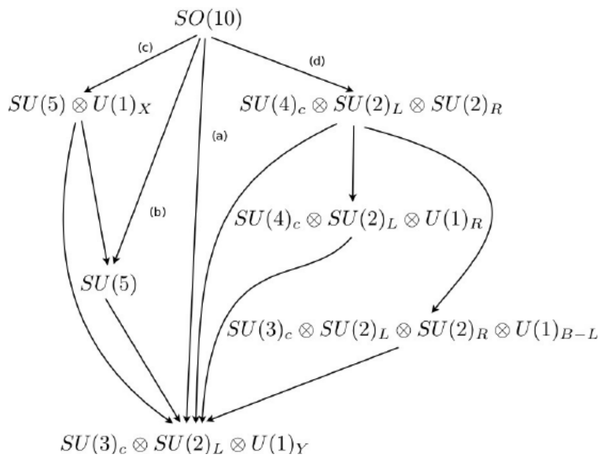


Figure 1.3: Symmetry breaking patterns of $SO(10)$ to $SU(3) \times SU(2) \times U(1)$; taken from [22]

The fundamental tensor representation of $SO(10)$, ϕ_α (with α running from 1 to 10), can be easily decomposed into representations of $SU(5)$ after forming the combinations $\psi^i = (\phi_1 + i\phi_6, \phi_2 + i\phi_7, ..., \phi_5 + i\phi_{10})$ and the complex conjugate $\overline{\psi}_i$. Since the squared modulus of ψ is the same as the squared modulus of ϕ, we find that $U(5)$ is a subgroup of $SO(10)$, and that ψ and $\overline{\psi}$ transform as a $\mathbf{5}$ and a $\overline{\mathbf{5}}$ respectively. By composing the representations, it is easy to see that the 45 generators of $SO(10)$ decompose into $SU(5)$ in $\mathbf{24}$ (which will be associated to the 24 generators of $SU(5)$), $\mathbf{1}$, $\mathbf{10}$ and $\overline{\mathbf{10}}$.

More complicated is the identification of the matter content of the Standard Model. This is to be found in the spinor representation of $SO(10)$; this representation is obtained by the observation that it is possible to construct 10 32×32 matrices Γ_j which satisfy the anticommutation relations:

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\delta_{\alpha\beta} \tag{1.26}$$

The matrices $\sigma_{\alpha\beta} = \frac{1}{2}[\Gamma_\alpha, \Gamma_\beta]$ now satisfy the same commutation relations as the orthogonal generators of $SO(10)$; it follows that the column spinor of 32 components which is acted upon by the Γ_j constitutes a representation of $SO(10)$. This representation is reducible; in fact, the operator $\Gamma_{11} = -\Gamma_1...\Gamma_{10}$ anticommutes with all other Γ_α. It follows that the space of spinors decomposes into eigenstates of Γ_{11} with positive and negative eigenvalues. We have thus found that $SO(10)$ admits irreducible $\mathbf{16}^+$ and $\mathbf{16}^-$ (according to the sign of the eigenvalue of Γ_{11}) representations.

Next question involves the decomposition of these representations into representations of $SU(5)$. To obtain these, notice that the generators of $SO(10)$, which we know how to decompose into representations of $SU(5)$, should transform $\mathbf{16}$ into itself. The 45 generators of $SO(10)$ include a singlet under $SU(5)$

and its 24 generators which leave the dimension of the $SU(5)$ representations composing the **16** unchanged. The other **10** and $\overline{\mathbf{10}}$ components are represented in $SU(5)$ as tensors with two lower or upper indices antisymmetrized; it follows that the **16** should decompose into representations of $SU(5)$ which differ from one another by a couple of antisymmetrical indices. It can then be seen that there are two possibilities: the $\mathbf{16}^+$ decomposes into **1**, $\overline{\mathbf{5}}$ and **10**, while the $\mathbf{16}^-$ into the complex conjugates. From our previous experience with $SU(5)$, we see that the matter fermions fit perfectly into the $\mathbf{16}^+$.

An useful way of looking at the decomposition of $SO(10)$ into $SU(5)$ is to notice that matrix elements between two spinors of Γ_μ transform as vectors of $SO(10)$; thus we can form the combination $\Gamma^i = (\Gamma_1 + i\Gamma_6, ...)$ and $\Gamma_i = (\Gamma_1 - i\Gamma_6, ...)$ (with i running from 1 to 5) which transform as a **5** and a $\overline{\mathbf{5}}$ under $SU(5)$. Then the spinor representation of $SO(10)$ can be written as:

$$\begin{aligned}\psi = \theta + (\Gamma_i\theta)\,\psi^i + (\Gamma_i\Gamma_j\theta)\,\phi^{ij} + (\Gamma_i\Gamma_j\Gamma_k\theta)\,\eta^{ijk} \\ + (\Gamma_i\Gamma_j\Gamma_k\Gamma_l\theta)\,\xi^{ijkl} + (\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m\theta)\,\zeta^{ijklm}\end{aligned} \tag{1.27}$$

which can be rewritten as:

$$\begin{aligned}\psi = \theta + (\Gamma_i\theta)\,\psi^i + (\Gamma_i\Gamma_j\theta)\,\phi^{ij} + (\Gamma_i\Gamma_j\Gamma_k\theta)\,\epsilon^{ijklm}\eta_{lm} \\ + (\Gamma_i\Gamma_j\Gamma_k\Gamma_l\theta)\,\epsilon^{ijklm}\xi_m + (\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m\theta)\,\epsilon^{ijklm}\zeta\end{aligned} \tag{1.28}$$

where θ is an $SO(10)$ spinor which behaves as a singlet under $SU(5)$. By dividing out the terms according to their behavior under Γ_{11} we find the decomposition which we have stated somewhat heuristically above. In particular, $\mathbf{16}^+$, into which the matter fields falls, decompose as:

$$\psi = \theta\nu_R + (\Gamma_i\Gamma_j\theta)\,\phi^{ij} + (\Gamma_i\Gamma_j\Gamma_k\Gamma_l\theta)\,\epsilon^{ijklm}\psi_m \tag{1.29}$$

where ϕ^{ij} and ψ_m are the matter fields of $SU(5)$ (the field which appears in the singlet term is explained below).

The $SU(5)$ singlet which appears in the $\mathbf{16}^+$ should represent a particle which does not interact through any of the Standard Model interaction; it is tempting to identify it with the right-handed neutrino of the seesaw mechanism. To complete this identification, we should discuss the fermion mass terms. The invariant $SO(10)$ combinations which can be constructed are the following:

$$Y^{ab}\psi_a^T C\Gamma_\mu\psi_b\phi^\mu, Y^{ab}\psi_a^T C\Gamma_\mu\Gamma_\nu\Gamma_\alpha\psi_b\phi^{\mu\nu\alpha}, Y^{ab}\psi_a^T C\Gamma_\mu\Gamma_\nu\Gamma_\alpha\Gamma_\beta\Gamma_\sigma\psi_b\phi^{\mu\nu\alpha\beta\sigma} \tag{1.30}$$

where a,b are flavor indices (the combinations with an even number of Γ_j vanishes because the spinors are eigenstates of Γ_{11}) and C is the charge conjugation operator. We see that the possible Higgs representations which give mass to the fermions are the **10**, the **120** and the **126**; in particular, since we are dealing with fermions, the terms need to be antisymmetric in the exchange of fields. It can now be proved that this implies that the matrix Y_{ab} is antisymmetric for the **120** and symmetric for the **10** and **126**.

The seesaw mechanism requires the presence of a Majorana mass term for the singlet of $SU(5)$; among the above three possibilities, only the **126** contains, in its decomposition into $SU(5)$ representations, a singlet which can give such a mass term.

Let us give an example of how one of the previous mass terms, the **10**, breaks the $SO(10)$ symmetry to $SU(5)$. Since the Standard Model group should stay unchanged, only the 5 and 10 components of the field ϕ^μ can acquire non vanishing expectation values; we can substitute (1.29) into the corresponding mass term. The problem reduces to the computation of matrix elements of products of Γ_j; without even doing the calculations we can infer by symmetry that the only non vanishing elements will be $\overline{\theta}\Gamma^i\Gamma_j\theta \propto \delta^i_j$, $\overline{\theta}\Gamma^i\Gamma^k\Gamma_j\Gamma_l\theta \propto \delta^i_l\delta^k_j - \delta^i_j\delta^k_l$ (together with all elements which contain an equal number of Γ^i and Γ_i) and $\overline{\theta}\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m\theta \propto \epsilon_{ijklm}$. An explicit calculation brings then to the following mass rules:

$$M_u = M_{D\nu}, \quad M_d = M_l \tag{1.31}$$

where M_u, $M_{D\nu}$, M_d and M_l are respectively the mass matrices (in flavor space) of the up quark generation, the Dirac neutrino mass terms, the down quark generation and the heavy leptons.

An analogous reasoning with the **126** leads to similar relations:

$$M_{D\nu} = -3M_u, \quad M_l = -3M_d \tag{1.32}$$

We will discuss the consequences of this mass relation in Chapter 3, in connection with the specific model of symmetry breaking which we have used in this work.

Let us also briefly describe the simplest way of breaking the $SO(10)$ symmetry to $SU(5)$, that is, the Higgs field needed to give mass to the gauge bosons. The generators in $SO(10)$ carry two antisymmetric indices, so that the simplest mass term which we can add to the Lagrangian is:

$$-\frac{1}{2}A_{ij}A_{lm}\Phi^{ijlm} \tag{1.33}$$

where A_{ij} are the gauge fields and Φ^{ijlm} is an Higgs field which evidently must be antisymmetric in the first and the second couple of indices and symmetric in the exchange of the two couples of indices with one another; the easiest choice is to take it to be antisymmetric under the exchange of any possible couple. This representation, which is a **210** in $SO(10)$, contains in its decomposition under $SU(5)$ a **24**, which we recall to be the representation needed to break $SU(5)$ symmetry to $SU(3) \times SU(2) \times U(1)$; it follows that this is the minimal choice of Higgs fields necessary to break down the $SO(10)$ symmetry.

1.4 Anomalies

An anomaly is generally defined as the breaking of a symmetry of a classical theory upon quantization [23, 24]. There are various ways in which the presence of an anomaly might be understood, so we will start by giving some heuristic notions about such a phenomenon.

Quantum field theory was actually faced with the presence of an anomaly since its beginning when it was understood that there was the necessity of a renormalization procedure. Let us look at the simplest classical field theory, the massless ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{4!}\lambda\phi^4 \tag{1.34}$$

It is evident that, under the set of transformations $x' = \alpha x$, $\phi' = \frac{\phi}{\alpha}$ the action stays unchanged. We therefore expect that a change of scale of length, or, equivalently, a scale of energy, should be a symmetry of the quantum theory as well as of the classical one. As is well known, however, on the quantum level the coupling constant λ is not constant but it depends on the scale of energy. On dimensional grounds, we have no parameter in the Lagrangian which has the dimensions of an energy, so there seems to be no correct dimensional way of expressing the dependence of λ on the scale of energy. The resolution of the paradox, of course, is that the instantaneous and pointlike interaction of the classical theory leads to divergences upon quantization, and must then be smoothed out through the introduction of a cutoff length scale (or, equivalently, a cutoff energy scale) which provides us with a parameter with the dimensions of an energy Λ. Scale invariance is then said to be anomalous, due to the fact that, in order to regularize the theory, one has to explicitly break such an invariance.

We can understand this breaking of scale invariance in a different way by looking at the path integral; the measure of the functional integral, in fact, is clearly not invariant under the transformation introduced above. We will show below that the non invariance of the functional measure in the path integral is the source of chiral anomaly as well as of scale invariance.

Chiral anomaly as pair production

We will show below that another anomaly is present on a quantum level. A simple way of understanding it is to notice that, in the presence of an electric field, electrons-positrons pairs can be produced[6]; if in the same region of space a magnetic field is present which is parallel to the electric field, pairs will be produced preferentially with the positron spin aligned along $\mathbf{B}$ and the electron spin antiparallel to it. This implies that chiral symmetry will be automatically broken by quantum fluctuations; our main task will be to derive mathematically this result.

Notice that this results depends critically on the infinities which are present in the quantum theory. The production of pairs, in fact, can be seen as the

[6]We can consider in this section the electromagnetic field to be classical; no changes are introduced in the relations which will be derived below

raising of a particle in the Fermi sea above the Fermi level; if the Fermi sea is bounded below, for each left-handed particle produced a left-handed antiparticle appears. The reason for the anomaly is that, in quantum field theory, the Fermi sea is unbounded below, so the appearance of a left-handed particle is not necessarily accompanied by the appearance of a left-handed antiparticle.

Chiral anomaly as deriving from gauge invariant regularization

Let us study the massless spinor electrodynamic theory defined by the following Lagrangian:

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - eA\!\!\!/\right)\psi \tag{1.35}$$

In an Hamiltonian context, the chiral current is defined as $j_5^\mu(x) = \overline{\psi}\gamma^5\gamma^\mu\psi$. This definition does not survive quantization in that it is ill defined [25]; the product of two operators at the same point of space is singular. In order to remove the singularity, we smooth out this definition by going to the limit $\epsilon \to 0$ of the gauge invariant expression:

$$j_5^\mu(x,\epsilon) = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^5\gamma^\mu e^{ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\alpha(x')dx'^\alpha}\psi\left(x-\frac{\epsilon}{2}\right) \tag{1.36}$$

For simplicity we will take the electron to be massless, so that when we differentiate this expression, after using the equations of motion and retaining only the first order term in ϵ we find:

$$\partial_\mu j_5^\mu(x,\epsilon) = -i\epsilon^\alpha e\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^5\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right)F_{\alpha\mu} \tag{1.37}$$

This is an exact expression, where we have taken the electromagnetic field to be classical. Next step involves averaging this over the vacuum state; in order to do this, we have to take the average of the Green function $\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^5\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right)$ for an electron in a given electromagnetic field over the vacuum state. If such a field is constant, we can exclude the possibility of the appearance of bound levels in the spectrum. We can then do this perturbatively to the first non vanishing order in e; a sketch of the calculation follows.

Because of translational invariance, the above Green function can be written as $\mathcal{G}^\mu(\epsilon) = \langle 0|\,T\left\{\overline{\psi}(\epsilon)\gamma^5\gamma^\mu\psi(0)\right\}|0\rangle$. We can find the differential equation which $\mathcal{G}^\mu$ has to satisfy; in a constant electromagnetic field $A^\mu(x) = \frac{F^{\mu\nu}x_\nu}{2}$:

$$i\partial\!\!\!/\mathcal{G}^\mu(\epsilon) + eA\!\!\!/(\epsilon)\mathcal{G}^\mu(\epsilon) = i\gamma^5\gamma^\mu\delta^4(\epsilon) \tag{1.38}$$

This can be solved perturbatively in e by the usual methods; another way is to write directly the expression in momentum space for $\mathcal{G}^\mu(p)$:

$$\mathcal{G}^\mu(p) = \gamma^5\gamma^\mu\frac{i}{p\!\!\!/ + eA\!\!\!/(p)} \tag{1.39}$$

The denominator of this expression can be expanded in e; after that, we need to take the trace of the operator. The first term whose trace with γ^5 does not vanish is:

$$Tr\left(\gamma^5\gamma^\mu \frac{i}{\not{p}}\left(-ie\not{A}(p)\right)\frac{i}{\not{p}}\right) \tag{1.40}$$

The trace can now be computed by the usual means to obtain:

$$\langle 0|\,\partial_\mu j_5^\mu\,|0\rangle = \frac{e^2}{4\pi^2}F_{\alpha\mu}F_{\beta\nu}\epsilon^{\lambda\mu\beta\nu}\frac{\epsilon_\lambda\epsilon^\alpha}{\epsilon^2} \tag{1.41}$$

In taking the limit for $\epsilon \to 0$ we have to take an average over direction because of spherical symmetry of space: the result is:

$$\langle 0|\,\partial_\mu j_5^\mu\,|0\rangle = \frac{e^2}{16\pi^2}F_{\mu\nu}F_{\alpha\beta}\epsilon^{\mu\nu\alpha\beta} \tag{1.42}$$

Chiral anomaly from Feynman diagrams

Historically chiral anomaly was first discovered through its consequences on the violation of Ward identities. (1.42), in fact, implies that some diagrams in massless fermion quantum electrodynamics will violate the Ward-Takahashi identity, even in the absence of an electromagnetic field. In fact, upon quantization of the electromagnetic field, from (1.42) we see that the divergence of the axial current is an operator capable of producing two photons. This implies that the so called triangle diagram in the computation of the matrix element $\langle 0|\,T\left\{j_5^\mu(x)j^\alpha(y)j^\beta(z)\right\}|0\rangle$ has a non vanishing divergence even in the absence of an electromagnetic field. This was confirmed by explicit computation of the corresponding Feynman diagram [17].

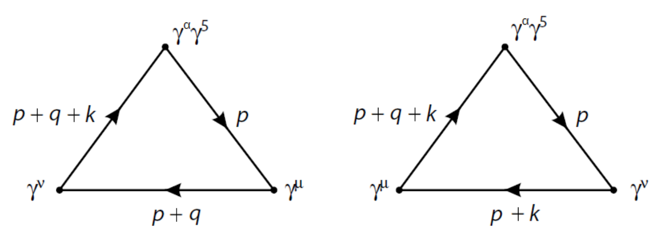


Figure 1.4: Triangle diagrams contributing to the matrix element $\langle 0|\,T\left\{j_5^\mu(x)j^\alpha(y)j^\beta(z)\right\}|0\rangle$

To first order in e^2 the matrix element above in momentum space corresponds to the sum of the two Feynman diagrams in Figure 1.4; by the usual rules this gives:

$$T_{\alpha\mu\nu} = Tr\left[\int\frac{d^4p}{2\pi^4}\left(\gamma^5\gamma_\alpha\frac{i}{\not{p}+\not{q}+\not{k}}\gamma_\nu\frac{i}{\not{p}+\not{q}}\gamma_\mu\frac{i}{\not{p}}+\gamma^5\gamma_\alpha\frac{i}{\not{p}+\not{q}+\not{k}}\gamma_\mu\frac{i}{\not{p}+\not{k}}\gamma_\nu\frac{i}{\not{p}}\right)\right] \tag{1.43}$$

If we want to find how the conservation laws are modified in this simple theory, we need to compute the three quantities $k^\nu T_{\alpha\mu\nu}$, $q^\mu T_{\alpha\mu\nu}$ (which correspond to the divergences of the vector currents j_μ in momentum space) and $(k^\alpha + q^\alpha)T_{\alpha\mu\nu}$ (which corresponds to the divergence of the axial current j^5_μ in momentum space). Let us give an example of how an integral of this form is computed.

By explicit substitution we find:

$$q^\mu T_{\alpha\mu\nu} = i\int \frac{d^4p}{2\pi^4} Tr\left[\gamma^5\gamma_\alpha \frac{i}{\not p + \not q + \not k}\gamma_\nu\left(\frac{i}{\not p} - \frac{i}{\not p + \not q}\right) + \gamma^5\gamma_\alpha\left(\frac{i}{\not p + \not k} - \frac{i}{\not p + \not q + \not k}\right)\gamma_\nu \frac{i}{\not p}\right] \tag{1.44}$$

After evaluating traces this becomes:

$$q^\mu T_{\alpha\mu\nu} = 4\epsilon_{\alpha\mu\nu\beta}\int \frac{d^4p}{(2\pi)^4}\left[\frac{(p^\mu + q^\mu + k^\nu)\left(p^\beta + k^\beta\right)}{(p+q+k)^2 (p+q)^2} - \frac{(p^\mu + k^\mu)\, p^\beta}{(p+k)^2\, p^2}\right] \tag{1.45}$$

The integrals are all linearly divergent; this means that they are not invariant under a translation of the integration variable. The question arises as to how we should choose the origin of the integration space to regularize the integral; the answer is that we should act in such a way as to preserve the symmetries of the classical theory. In particular, those symmetries which will be gauged (which in this case are associated with the vector current) are required to be conserved at the quantum level for the consistency of the theory; were they not conserved, in fact, the theory would not be renormalizable anymore.

Let us first study what happens to the integral of a linearly divergent function when we shift the integration variable:

$$\int d^4p\,[f(p+a) - f(p)] = \int d^4p\,[\partial_\mu f a^\mu + ...] = \int d\Sigma_\mu f(p) a^\mu \tag{1.46}$$

where Σ_μ is a surface at infinity. In 4 dimensions we can then write:

$$\int d^4p\,[f(p+a) - f(p)] = 2i\pi^2 a^\mu \lim_{P\to\infty} P^3 f(P)\,\frac{P_\mu}{P} \tag{1.47}$$

We have implicitly rotated to Euclidean space to write the volume of the compact hypersphere and then Wick rotated back to Minkowski space, which explains the origin of the i. For a linearly divergent integral the above limit is finite.

Our previous reasoning implies that the question of wheter (1.45) vanishes is ill posed, since it depends on our choice of the origin of the integration space. We can then come back to our original definition (1.43) and generalize it to compute $T_{\alpha\mu\nu}(a)$ where the integration variable is shifted by the 4-vector a. The difference between the two definitions can be easily written down using (1.47):

$$T_{\alpha\mu\nu}(a) - T_{\alpha\mu\nu} = \frac{i}{8\pi^2}\epsilon_{\alpha\mu\nu\beta}a^\beta + \{\mu, k \leftrightarrow \nu, q\} \tag{1.48}$$

We have maintained that k and q must be exchanged because a is generally dependent on k and q.

By explicitly computing $T_{\alpha\mu\nu}$ we can now find a such that the divergence of the vector currents vanishes; we find:

$$a_\mu = \alpha\left(k_\mu + q_\mu\right) - \frac{1}{2}\left(k_\mu - q_\mu\right) \tag{1.49}$$

where α is undetermined.

If we now try to compute the divergence of the axial current we find that it does not vanish and that it is equal to:

$$\left(k^\alpha + q^\alpha\right) T_{\alpha\mu\nu} = \frac{i}{2\pi^2}\epsilon_{\mu\nu\alpha\sigma}k^\alpha q^\sigma \tag{1.50}$$

Importantly, when we gauge our theory, the vector currents can be substituted by external photon lines, and we obtain once again (1.42).

The source of the anomaly can be traced back to the introduction of regulators which are not explicitly chirally invariant. For example, we recall that the Pauli-Villars regularization procedure requires the introduction of a ghost particle with a mass which is then taken to be infinite after the renormalization procedure; but the introduction of massive fermions explicitly spoils chiral invariance.

Chiral anomaly from path integrals

The source of the anomaly is particularly clear in the path integrals viewpoint [26]. We will deal with Euclidean path integrals, obtained after the substitution $t \to it$; thus the Dirac operator $\not{D} = \not{\partial} + ie\not{A}$ is hermitean and admits a set of eigenfunctions ϕ_k with eigenvalues λ_k.

Under the transformation $\psi' = e^{i\gamma^5\alpha}\psi$, the classical action stays unchanged because of the chiral invariance of the theory, while the integration measure changes by a factor $e^{2\alpha\int d^4x\delta(x-x)Tr\left\{\gamma^5\right\}}$. This factor is clearly ill defined, and it needs to be regulated. We can substitute the integration over x and the trace by a sum over all the eigenfunctions of the gauge covariant Dirac derivative and insert a regulating function to obtain that the factor is $e^{\alpha\int d^4x\mathcal{A}(x)}$, where $\mathcal{A}(x)$ is the anomaly function defined by:

$$\mathcal{A}(x) = 2\sum_k\left(\phi_k^*\gamma^5 f\left(\frac{\lambda_k^2}{M^2}\right)\phi_k\right) \tag{1.51}$$

in the limit for the regularization scale $M \to \infty$. As $|k| \to \infty$ f must drop smoothly from 1 to 0; since we are not interested in proving the independence of the result on our choice, we will simply take $f = e^{-\frac{\lambda_k^2}{M^2}} = e^{-\frac{\not{D}_k^2}{M^2}}$. After the introduction of the regulating function, we can go back to the representation in terms of integral over x and trace over spinor index. The operator $\not{D}^2 = (\partial - ieA)^2 - \frac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu}$ appears in the exponent; we can now expand the exponential in terms of e; since the trace of γ^5 is non zero when it is contracted with 4 γ, it is clear that the first non vanishing contribution will appear from the term:

$$\mathcal{A}(x) = \frac{e^2}{4M^2}\left(\int d^4x e^{-\frac{\partial^2}{M^2}}\right) Tr\left\{\gamma^5\sigma_{\mu\nu}\sigma_{\alpha\beta}\right\} F^{\mu\nu}F^{\alpha\beta} \tag{1.52}$$

The integral over dx can be substituted by an integral over the momentum of free particle wavefunctions, which are eigenfunctions of the operator ∂^2; the integral can then be done easily and, in the limit $M \to \infty$ one obtains:

$$\mathcal{A}(x) = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{1.53}$$

The existence of the anomaly function implies that the effective action will not be invariant under a chiral transformation, and thus brings us back to (1.42).

All of the previous results have been derived in a one-loop approximation. However, Adler and Bardeen [27] proved that the form of the anomaly is not subject to higher order corrections. A simple way to see this is to note that the integral of the left-hand side of (1.42) over all space gives the rate of change of left-handed minus right-handed particles, which must be an integer; this implies that both sides of the equation must change only by integer quantities.

Anomalies in the Standard Model

In non-abelian gauge theories the amplitude for the triangle diagrams which violate the Ward-Takahashi identities contains a factor $Tr\left\{T_a\left\{T_b, T_c\right\}\right\}$. Since in the Standard Model left-handed particles couple differently to the $SU(2)$ generators than right-handed particles, it might be expected that currents associated to gauge fields might be anomalous. This would however create problems in the quantization of the theory; luckily, in the Standard Model these currents turn out not to be anomalous. We can prove the more general result that any theory which can be deduced from spontaneous symmetry breaking in $SO(10)$ (actually in $SO(2n)$ with $n \neq 3$) has gauge currents which are free from anomaly [17]. In fact, the generators in $SO(10)$ brings into the anomaly a factor of $Tr\left\{J_{mn}\left\{J_{ik}, J_{st}\right\}\right\}$; but in $SO(10)$ there is no invariant tensor with 6 indices, so it follows that this factor is 0.

The reasoning above is valid, of course, only for the currents associated to gauge fields, since we had to sum over all fermion representations. In the Standard Model, the currents associated to baryon and lepton number turn out to be anomalous:

$$\partial_\mu j^\mu_B = -\partial_\mu j^\mu_L = 3\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}W^a_{\mu\nu}W^a_{\alpha\beta} \tag{1.54}$$

where W^a are the $SU(2)$ gauge fields; the factor of 3 comes from the number of different generations in the Standard Model.

1.5 Instantons and topological processes

The anomalous violation of conservation laws suggests the possibility of the existence of processes which explicitly violate the conservation of baryon and lepton numbers. It is clear that no such process will be observed at any order of the perturbative treatment given by Feynman diagrams; by its very nature, in fact, perturbation theory studies small fluctuations around a vacuum state (which we have taken to be the state with no particles in the Fock space). Since our classical theory preserves chiral symmetry, the Feynman rules will always conserve baryon and lepton numbers.

Therefore, we expect the effects of anomaly to be present only on a non perturbative level. We already know that perturbation theory fails in dealing with tunneling effects between different vacua; a specific example of this has been given in the case of spontaneous symmetry breaking. In that case, we restricted ourselves to a specific configuration which spontaneously breaks the symmetry of the Lagrangian and neglected the possible transitions between different such configurations. Of course this cannot be done in the context of perturbation theory; the ideal means of dealing with this problem is the WKB approximation.

Tunneling probabilities in quantum field theory

As was derived for the first time from Landau [28], the transition probability between two states separated by an energy barrier like in Figure 1.5 is proportional to the exponential of the classical action over the complex trajectories[7] which connect the two states. The factors which appear together with the exponential can be derived in a simpler way in a path integral context.

The tunneling amplitude is given by the expression:

$$\mathcal{K}\left[\phi_b, \phi_a, t\right] = \int \mathcal{D}\phi e^{iS[\phi]} \tag{1.55}$$

where ϕ is the set of fields present in our theory, $S\left[\phi\right]$ is the action and the integral is taken over the paths which connect the two classical configurations of interest ϕ_a and ϕ_b over the time t.

We expect the integral to be dominated by paths which make the exponent stationary; if the two extremal configurations are not classically connected (which is the case of interest since we want to study tunneling) then no such path exist. It is possible, however, to find configurations of fields which extremize the exponent for complex values of t. Such configurations give the most important contributions to the integral; these contributions are, as is easily seen, exponentially damped (due to the imaginary time in (1.55)) by a factor e^{-S_0}, where S_0 is the action evaluated over these classical paths. We can now evaluate the accompanying factors by explicitly doing the integrals over small

[7]There are no real trajectories since transitions between the two states are classically prohibited.

fluctuations around these classical solutions; this is done in detail for a simple quantum mechanical case, for example, in [29]. The most important conclusion which we draw is that tunneling effects can happen in quantum field theory and that their amplitude is suppressed by a factor $\sim e^{-S_0}$.

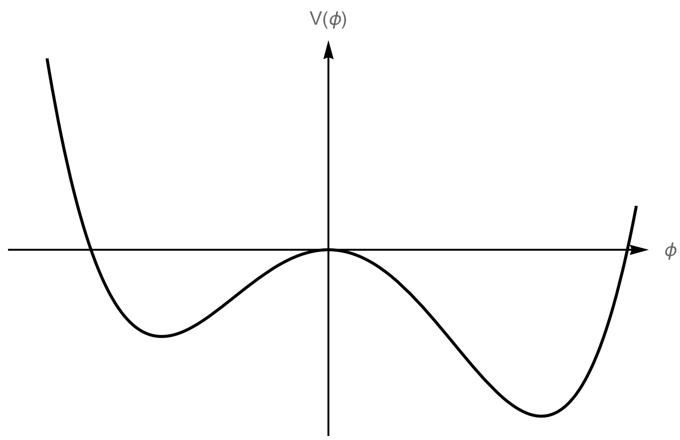


Figure 1.5: Model potential for barrier penetration in a scalar field theory

We will also need an estimate of the rate of such processes at finite temperatures. A first complication which we encounter is the definition of the decay rate of a metastable state at finite temperature. Suppose we have two sets of field configurations separated by a potential barrier; a model example which can be used is the case of a single scalar field with a potential energy having two minima which can be taken to be different from one another like in Figure 1.5. We can then define the probability that, after a time t, the field tunnels from one side of the potential barrier to the other as:

$$\int \mathcal{D}\phi_a P\left[\phi_a\right] \int \mathcal{D}\phi_b \left|\langle\phi_b| e^{-iHt} |\phi_a\rangle\right|^2 \tag{1.56}$$

In (1.56) the initial configuration ϕ_a is thermally averaged over all *static* configurations belonging to the side of interest of the potential barrier, while the final configuration ϕ_b is summed over all static configurations belonging to the other side of the barrier. $P\left[\phi_a\right]$ is chosen to be the thermal distribution function $\sim e^{-\beta H}$ *projected over the side of interest of the barrier*, since we are interested in the decay rate of states belonging to one minimum of the potential. This expression is expected to depend on time as $\sim (1 - e^{-\Gamma t}) \sim \Gamma t$ where Γ is the decay rate and we have expanded for $t \ll \Gamma^{-1}$.

The expression (1.56) can be further reduced to the form:

$$\frac{1}{\mathcal{Z}} \int \mathcal{D}\phi_a \int \mathcal{D}\phi_b \, \langle\phi_b| e^{-iHt} |\phi_a\rangle \, \langle\phi_a| e^{-\beta H} |\phi_a\rangle \, \langle\phi_a| e^{iHt} |\phi_b\rangle \tag{1.57}$$

where $\mathcal{Z}$ is the partition function restricted to the first side of the barrier. The real time matrix elements are easily calculated at zero temperature; they collectively give, for $t \ll \Gamma^{-1}$, a factor:

$$D[\phi_a] = \int \mathcal{D}\phi_b e^{-2S_0[\phi_b,\phi_a]} \tag{1.58}$$

where S_0 is defined as above and ϕ_b is integrated over all configurations on the final side of the barrier. The integration will be dominated by a stationary point $\overline{\phi}_b$, which is basically the point on the final side of the barrier with the same potential energy as ϕ_a, so that we can write $D[\phi_a] \sim e^{-\overline{S}[\phi_a]}$ where $\overline{S}[\phi_a] = S_0\left[\overline{\phi}_b, \phi_a\right]$.

The thermal matrix element can be rewritten as a path integral:

$$\int \mathcal{D}\overline{\phi} e^{-\int_0^\beta du H\left[\overline{\phi}\right]} \tag{1.59}$$

where $\overline{\phi}$ are *dynamical* paths in ϕ space which brings configuration ϕ_a back to itself at "time" β. For high temperatures, a well known approximation [30] is to compute only the path which stays fixed at configuration ϕ_a and fluctuations around it; then the integral will give a factor $e^{-\beta V[\phi_a]}$, where V is the potential energy of the Hamiltonian.

We will finally have to integrate the expression:

$$\int \mathcal{D}\phi_a e^{-\beta V[\phi_a]} e^{-2S_0[\phi_a]} \tag{1.60}$$

over configurations ϕ_a belonging to the initial side of the barrier. The integration will be dominated by static configurations which extremize the integral; for high temperatures, the action $S_0[\phi_a]$ is clearly minimized by points near the maximum of the potential barrier, where $S_0[\phi_a] \sim 0$. This implies that the thermal decay rate is suppressed by a factor $\sim e^{-\beta V[\Phi]}$, where Φ is the unstable static configuration corresponding to the maximum of the potential barrier.

Instantons and sphalerons in gauge field theory

We will now apply the general theory developed until now to gauge field theory. For gauge fields the action, after rescaling $A \to \frac{A}{g}$, is:

$$S = -\frac{1}{2g^2} \int d^4x F^a_{\mu\nu} F^{a\mu\nu} \tag{1.61}$$

We look for stationary points of this action; in particular, we will look for classical *dynamical* solutions to describe tunneling at zero temperature and for classical *static unstable* solutions to describe tunneling at high temperatures. Let us start with the former, which are known in literature as *instantons* [29].

The solutions of interest are to be found among those of finite action; in fact, WKB approximation around solutions of infinite action gives vanishing contribution. This implies that at spatial infinity, for the initial and final time, the solution should approach a pure gauge; we therefore need to solve the classical field equations with boundary conditions corresponding to a pure gauge field to find the initial and final configuration, and then find a dynamical solution which connect them. An objection might be raised that if we try to solve the gauge invariant field equations with such boundary condition we might find only solutions which are *everywhere* gauge equivalent to zero field. The solution of the paradox lies in the fact that the gauge transformation needs to be singular only at spatial infinity, while it can have singularities in the interior of space. Two configurations which are related at spatial infinity by a gauge transformation which is regular *everywhere* are said to be homotopically equivalent (since they admit the same topological mapping of the gauge group to the spatial boundary); we can then order the static gauge field configurations into equivalence classes. The tunneling solution will now connect two static field configurations belonging to two different classes. The action for the field can be rewritten in the form:

$$S = -\frac{1}{4g^2} \int d^4x \left[\left(F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu} \right)^2 + 2F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \right] \tag{1.62}$$

where:

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{a\alpha\beta} \tag{1.63}$$

The last term, which is identical in form to the so called Chern-Simons term, consists of a pure divergence:

$$F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = \partial_\mu K^\mu \tag{1.64}$$

where:

$$K_\mu = 2\epsilon_{\mu\nu\alpha\beta} A^{a\nu} \left(\partial^\alpha A^{a\beta} - \frac{i}{3} f^{abc} A^{b\alpha} A^{c\beta} \right) \tag{1.65}$$

where f^{abc} are the antihermitean structure constants defined by:

$$\left[t^a, t^b \right] = f^{abc} t^c \tag{1.66}$$

This last term does not contribute to the dynamical equations of motion; the first term is clearly minimized by the choice:

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} \tag{1.67}$$

We are interested in solutions for imaginary time; this might create some ambiguities in the definition of A^a_0, which should be chosen to be imaginary. We can however avoid this subtlety by choosing a gauge in which A^a_0 vanishes. The tunneling amplitude now is suppressed by a factor e^{-S_0} where:

$$S_0 = \frac{1}{2g^2} \int d^4x F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = \frac{1}{2g^2} \int d\Sigma^\mu K_\mu \tag{1.68}$$

Since we are studying tunneling solutions, the integration surface can be chosen to consist of the union of two 3-surfaces at $t \to \pm\infty$, so that the previous integral will be written as the difference of two integrals:

$$Q_{\pm} = \frac{1}{2g^2} \int d\Sigma^4_{\pm} K_4 \tag{1.69}$$

where the fourth component is identified with Euclidean time.

Since the gauge field vanishes at spatial infinity only the second term of (1.65) contributes; furthermore, the fact that S_0 depends on A^a evaluated at infinity, where it is a pure gauge, implies that the result will depend only on the homotopic class to which the classical solution belongs. By substituting the gauge form $A^b_\mu = \partial_\mu \theta^b(x)$, (1.69) becomes:

$$-\frac{i}{3g^2} \int d^3x \epsilon_{ijk} f^{abc} \partial_i \theta^a \partial_j \theta^b \partial_k \theta^c \tag{1.70}$$

For $SU(2)$ the map $x^i \to \theta^a$ can be regarded as a topological mapping of real space onto the group $SU(2)$; we can also substitute the structure constants for $SU(2)$:

$$+\frac{2}{3g^2} \int d^3x \epsilon_{ijk} \epsilon^{abc} \partial_i \theta^a \partial_j \theta^b \partial_k \theta^c \tag{1.71}$$

By using the properties of the Levi-Civita symbols and recognizing the Jacobian determinant of the above mentioned map we can rewrite the integral in the form:

$$-\frac{4}{g^2} \int d^3\theta \tag{1.72}$$

This integral is now the volume in θ space covered by the topological map. For the simplest case that θ^a is exactly proportional to $\frac{x^a}{\sqrt{x^a x_a}}$, which corresponds to the identity mapping of the physical 3-sphere with the $SU(2)$ group, the integral will then give simply the solid angle in 4 dimensions and we find:

$$Q = \frac{8\pi^2}{g^2} \tag{1.73}$$

Members of different homotopic classes, as mentioned above, possess singularities of different orders; this means that we can compose the simple gauge transformation given above to obtain all other homotopic classes. The result is that for the map obtained by the composition of $n_{\pm}$ simple map of the form given above:

$$Q_{\pm} = \frac{8\pi^2 n_{\pm}}{g^2} \tag{1.74}$$

The integer number $n_{\pm}$ is known as the winding number of the spatial configuration at $t \to \pm\infty$; it measures the number of times the angles θ_a, in their mapping in physical space, are wrapped around the origin. The tunneling processes we are describing, therefore, happens between spaces with different winding numbers, and it is suppressed by a factor $\sim e^{-\frac{8\pi^2\nu}{g^2}}$, where $\nu = n_+ - n_-$.

In order to understand the physical meaning of the processes whose existence has now been proven, we observe that, apart from constant factors, the Euclidean action which we have calculated appears in the anomaly expression (1.54); by integrating that equation over all spacetime in the same way as we did before we find that the initial and final state of the tunneling process, which have winding numbers differing by an integer ν, possess baryon and lepton numbers which differ by 3ν and -3ν respectively[8]. The conclusion we draw is that there exist in the Standard Model baryon and lepton number violating processes, which are, however, strongly suppressed at zero temperature.

At high temperatures, by the general theory introduced above, we need to identify static unstable configurations of gauge fields; such configurations are generally known as *sphalerons* [31]. In the case of massless gauge fields they can take arbitrarily low values of energy; a simple way of proving this is to note that the gauge field Hamiltonian in 3 dimensions:

$$H\left[A^{\mu}\right] = -\int d^3x \frac{1}{2g^2} F_{ij}F^{ij} \tag{1.75}$$

under the transformation $A \to \lambda A(\lambda x)$, $E \to \lambda E$. For any given configuration we can then choose λ in such a way as to lower the energy; this means that the height of the energy barrier vanishes. As a consequence, in pure gauge theory thermal tunneling is not suppressed at all.

In the Standard Model, however, gauge fields become massive through the Higgs mechanism; it is clear, therefore, that the sphaleron solution should be looked for in the context of a gauge field endowed with a Higgs field.

The existence of these solutions has been proven in [32]; by computing the energy of these configurations we can find the thermal suppressing factor above defined. The results of this calculation have been used to find [33] that at temperatures above about 300 GeV the rate of these processes is comparable with the rate of expansion of the Universe (this will be discussed in more detail in Chapter 2) so that the sphaleron reactions are at equilibrium.

[8] As an aside we notice that the fact that the two members of this equality are both integers implies that this equation holds to all orders of perturbation theory, which constitutes the so called non renormalization of the anomaly.

Chapter 2

Thermal history of the Universe and Baryogenesis

2.1 Friedmann-Robertson-Walker metric

The description of the Universe on a large scale, in a general relativistic context, requires the establishment of a spacetime metric; the connection between this and the matter content is specified by the field equations of general relativity. In order to solve them, we need to introduce some assumptions on the geometry of spacetime.

The metric which will be assumed to describe our Universe is the Friedmann-Robertson-Walker metric [34]. This is obtained under the assumption of spatial homogeneity and isotropy of the Universe. We can obtain the most general metric satisfying these requirements by the imposition of two equivalent constraints; either we require the spatial curvature, which is the trace of the 3-dimensional Ricci tensor, to be constant, or we require that the spatial metric admits 6 Killing vectors, corresponding to 3 translations and 3 rotations. By imposing either of these constraints[1], one obtains the following form for the 3-dimensional metric:

$$dl^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \tag{2.1}$$

where $d\Omega^2$ is the angular part of the metric in polar coordinates and k is a parameter which can take the values $+1$, 0 and -1. Turning to the spacetime metric, we see that, in order to preserve homogeneity and isotropy at each moment of time, we can choose a synchronous system of coordinates in which:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{2.2}$$

[1]There is another possibility which consists in explicitly imposing the geometrical constraint of homogeneity and isotropy on the length of segments in this spacetime; this proof can be found in [35].

In (2.2) $a(t)$ is the time dependent scale function, which sets the scale of distances in the Universe; its time evolution is specified by the Einstein field equations (which in this context are known as Friedmann equations):

$$H^2(t) + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.3}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and ρ and p are the matter density and pressure (which by homogeneity are constant everywhere). They include contributions not only from ordinary matter but also from vacuum energy; this might lead to a term in the Friedmann equations known as the cosmological constant term (the name derives from the fact that it can be obtained from the addition to the Einstein equations of a term $\Lambda g_{\mu\nu}$).

In order to solve the Friedmann equations, one has to specify an equation of state, which requires a specification of the matter content of the Universe. In general each of the components of the total energy density of the Universe possesses a specific equation of state; the usual simplifying assumption is made that non-relativistic particles behave as dust with zero pressure[2] and relativistic particles behave as zero mass particles with $p = \frac{\rho}{3}$. The cosmological constant term which was mentioned before has, as it can be clearly seen, an equation of state of the form $p = -\rho$.

The role played by each of the above components (which we will refer to as matter, radiation and cosmological constant components respectively) can be conveniently represented through the dimensionless parameter:

$$\Omega_j = \frac{\rho_j}{\rho_c} \tag{2.4}$$

where:

$$\rho_c = \frac{3H^2(t)}{8\pi G} \tag{2.5}$$

is the so called critical density. Let us also introduce a curvature density and pressure:

$$\rho_k = -\frac{3k}{8\pi G a^2}, \quad p_k = -\frac{\rho}{3} \tag{2.6}$$

chosen in such a way that the Friedmann equations can be rewritten as:

$$\sum_j \Omega_j = 1, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_j (\rho_j + 3p_j) \tag{2.7}$$

The second of Friedmann equations can be rewritten as a statement on the evolution of density with time; it can be easily proved that for dust, radiation,

[2]It can be proved that pressure is smaller than energy density by a factor $\sim \frac{m}{T}$ where m is the mass of the particle and T is the temperature of the species.

cosmological constant and curvature we have respectively:

$$\rho_{dust} \sim \frac{1}{a^3}, \quad \rho_{rad} \sim \frac{1}{a^4}, \quad \rho_\Lambda \sim const., \quad \rho_k \sim \frac{1}{a^2} \tag{2.8}$$

Since observational data show that a is increasing with time, we see that the matter and radiation contribution decreases with time more rapidly than the Λ one; at the present time $\Omega_\Lambda \sim 0.7$ and $\Omega_{dust} \sim 0.3$, while radiation gives a negligible contribution ($\Omega_{rad} \sim 10^{-5}$). Going backward in time we see that the cosmological constant contribution was less and less important while radiation and matter gave increasing contributions until an equivalence point at which these two components gave exactly the same contribution. Before that moment, the Universe was dominated by radiation and massless particles.

The curvature contribution is today estimated to be extremely small; moreover, it can be proved that, under the assumption of decelerated expansion, $k = 0$ is an unstable fixed point of the evolution of Friedmann equations. This implies that either k is exactly 0 or that its effect had to be extremely small at the beginning of the Universe. This so called flatness problem has been solved by inflation, according to which the curvature contribution to the energy density of the Universe was strongly suppressed by an initial period of accelerated expansion; the mathematical details are of no interest here. We will simply note that, in our discussion of baryogenesis, we can put $k = 0$ in the Friedmann equations, since its contribution is unimportant at the time of baryogenesis (which is successive to inflation).

2.2 Thermodynamics in the expanding Universe

The next two sections will be devoted to a brief exposition of thermodynamics and kinetic theory in the expanding Universe. We will begin with equilibrium thermodynamics [35].

A given species of particles in the Universe will be characterized by its number density n_j, its energy density ρ_j, its pressure p_j, its chemical potential μ_j, its temperature T_j and its entropy density s_j (by the capital letters we will denote the total quantity, which is the density multiplied by the volume because of homogeneity). A fundamental link between these two quantities is furnished by the first principle of thermodynamics:

$$dU_j = T_j dS_j - p_j dV + \mu_j dN_j \tag{2.9}$$

By substituting the total quantities in terms of their densities we find:

$$dV\left(\rho_j - T_j s_j + p_j - \mu_j n_j\right) + V\left(d\rho_j - T_j ds_j - \mu_j dn_j\right) = 0 \tag{2.10}$$

Since the variations of extensive and intensive quantities are independent, we can simply set to 0 the coefficient of dV, obtaining:

$$s_j = \frac{\rho_j + p_j - \mu_j n_j}{T_j} \tag{2.11}$$

The first principle then reads simply:

$$d\rho_j = T_j ds_j + \mu_j dn_j \tag{2.12}$$

We have talked until now about the temperature of the species without caring much about how this thermal equilibrium is established. As long as interaction rates are comparable to the rate of expansion of the Universe we expect the jth species to stay in equilibrium with the other species at a fixed temperature which we might call the temperature of the bath which constitutes the Universe; when interaction rates become negligible with respect to the rate of expansion of the Universe, we can simply consider the species at equilibrium at a time dependent temperature which is determined only by the Universe expansion[3]. The intermediate phase cannot of course be described in an equilibrium context.

To further specify the evolution we need to relate the pressure and the number density of the species with its energy density; this requires of course a specific statistical model. Thus we will recall some basic facts about thermodynamic distributions.

Relativistic quantum theory shows that, in order to grant the existence of a stable ground state, particles with integer spin should be quantized with commutation rules between creation and annihilation operators while particles with half-integer spin should be quantized with anticommutation rules; this is generally known as the spin-statistics connection. For non interacting particles we can define energy levels; then the spin-statistics connection implies that each level can be occupied by any number of particles in the bosonic case and by at most a single particle in the fermionic case. The expected number of particles for each energetic level ϵ_j is then given in the two cases by:

$$N_j = \frac{1}{e^{\beta(\epsilon_j - \mu)} \pm 1} \tag{2.13}$$

where the $+$ is for fermions and the $-$ for bosons. We can now sum over all states to obtain the mean number density and mean energy density; in the sum over energetic states we can pass to the continuum limit by introducing the integral:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} g_j \tag{2.14}$$

where g_j is the multiplicity factor which takes into account the discrete number of states of each species (for example, for photons $g_j = 2$, corresponding to the two orthogonal polarization states). Then we can write the expressions

[3]This is rigorously true for relativistic species, as can be proven through the Boltzmann equation introduced below; for non relativistic species this constitutes only an approximate solution.

corresponding to the number density, energy density and pressure:

$$n_j = \int \frac{d^3\mathbf{p}}{(2\pi)^3} g_j \frac{1}{e^{\beta(\epsilon(\mathbf{p})-\mu)} \pm 1} \tag{2.15}$$

$$\rho_j = \int \frac{d^3\mathbf{p}}{(2\pi)^3} g_j \frac{1}{e^{\beta(\epsilon(\mathbf{p})-\mu)} \pm 1} \epsilon(\mathbf{p}) \tag{2.16}$$

$$p_j = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} g_j \frac{1}{e^{\beta(\epsilon(\mathbf{p})-\mu)} \pm 1} \frac{\mathbf{p}^2}{\epsilon(\mathbf{p})} \tag{2.17}$$

where the expression for the pressure has been derived by standard kinetic arguments; it is the flux of impulse (that is, the impulse $\mathbf{p}$ multiplied by the velocity of the particle $\frac{\mathbf{p}}{E}$) with a factor $\frac{1}{3}$ coming from an average over directions.

For high enough temperature both Bose-Einstein and Fermi-Dirac distributions behave as the Maxwell-Boltzmann distribution; in this case the integrals can be written in terms of modified Bessel functions. The condition of applicability is that the chemical potential is much smaller than the typical energy of the occupied levels; since such energy is of order $\sim T_j$, this requires that $\mu_j \ll T_j$.

For the particle number density we have:

$$n_j = \frac{g_j M_j^2 T_j}{2\pi^2} K_2\left(\frac{M_j}{T_j}\right) \tag{2.18}$$

where M is the mass of the particle and $K_2(z)$ is the modified Bessel function of the second kind.

It will also be necessary to have an expression for the jth species contribution to the total pressure, entropy and energy density of the Universe. We first note that non relativistic particles give a contribution which is exponentially smaller (by the Boltzmann factor $e^{-\frac{M_j}{T_j}}$) than relativistic ones, so that we can consider only the latter. In the relativistic limit the integrals can be done exactly; we will also use the approximation that $T_j \gg \mu_j$, which is true at the temperatures of interest. The results are:

$$n_j = \frac{\zeta(3)}{\pi^2} g_j T^3, \quad \rho_j = \frac{\pi^2}{30} g_j T_j^4, \quad p_j = \frac{\rho_j}{3} \tag{2.19}$$

for bosons and:

$$n_j = \frac{3\zeta(3)}{4\pi^2} g_j T^3, \quad \rho_j = \frac{7}{8}\frac{\pi^2}{30} g_j T_j^4, \quad p_j = \frac{\rho_j}{3} \tag{2.20}$$

for fermions.

The total entropy density and the total energy density, then, through (2.11), can be written as:

$$\rho = \frac{\pi^2}{30} g^* T^4, \quad s = \frac{2\pi^2}{45} g_S^* T^3 \tag{2.21}$$

where T is the mean temperature of the thermal bath of the Universe and g^* and g^*_S are effective degrees of freedom defined by:

$$g^* = \sum_{bosons} g_j \left(\frac{T_j}{T}\right)^4 + \frac{7}{8} \sum_{fermions} g_j \left(\frac{T_j}{T}\right)^4 \tag{2.22}$$

and:

$$g^*_S = \sum_{bosons} g_j \left(\frac{T_j}{T}\right)^3 + \frac{7}{8} \sum_{fermions} g_j \left(\frac{T_j}{T}\right)^3 \tag{2.23}$$

These factors will depend on whether the jth species is at equilibrium with the thermal bath, and thus will be strongly dependent on the particular moment of the evolution of the Universe which we are dealing with.

As long as the chemical potential of the distribution vanishes, no asymmetry is produced in the species; since we will be interested in describing the generation of asymmetry in various species, we will now derive a simple relation between the chemical potential and the asymmetry of a species for massless particles (since this is the regime which will be of interest to us) and for $\mu \ll T$.

The starting point is the observation that photons have vanishing chemical potential; this can be easily derived from equilibrium conditions imposed on the reaction $e \rightarrow e + \gamma$. Since particles and antiparticles can annihilate into a photon, it follows that at equilibrium, for each species X, $\mu_X = -\mu_{\overline{X}}$ (if these particles do not annihilate electromagnetically, the same result can be derived from their annihilation reaction into other gauge bosons). The form (2.15) of the distribution implies:

$$n_X - n_{\overline{X}} = 2g \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{\beta p} \sinh \beta\mu}{e^{2\beta p} \pm 2e^{\beta p} \cosh \beta\mu + 1} \tag{2.24}$$

By expanding with respect to $\beta\mu$ we deduce:

$$n_X - n_{\overline{X}} = \frac{g\mu T^2}{3} \tag{2.25}$$

for bosons and:

$$n_X - n_{\overline{X}} = \frac{g\mu T^2}{6} \tag{2.26}$$

for fermions. If we define the asymmetry of a given species as $\mathcal{X} = \frac{n_X - n_{\overline{X}}}{n_X}$, to first order in $\frac{\mu}{T}$ we find:

$$\mathcal{X} = \frac{\pi^2 \mu}{3\zeta(3)T} \tag{2.27}$$

for bosons and:

$$\mathcal{X} = \frac{2\pi^2 \mu}{9\zeta(3)T} \tag{2.28}$$

for fermions. The use of this relations is not much in numerical computation; they instead serve to show us that, at thermal equilibrium, asymmetries obey the same relations as chemical potentials.

2.3 Kinetic theory in the expanding Universe

Kinetic theory is based upon the introduction of a distribution function in the particles phase space. Rigorously speaking such a distribution function will depend over the spacetime coordinates and momenta of each particle. It is clear that no simplification is obtained in dealing with a function of $8N$ variables (N being the number of particles) instead of explicitly solving the evolution equations for each such particle.

The usual approximation consists then in integrating this function over the coordinates of all particles but one, and obtaining in this way a one-particle distribution function $f(x^\mu, p^\nu)$. This inevitably leads to a loss of information about correlations between particles; in particular, such an approximation will force us to expose a derivation of the evolution equations for the distribution function f which can be regarded as only heuristic in nature and not completely rigorous.

Before deriving such an evolution equation (which is known in literature as Boltzmann equation) we have to deal with the subtleties introduced by the spacetime metric [36]. Because of homogeneity and isotropy f can only depend on time and on the energy of the particle. If the species is not subject to any interaction, then its distribution function should admit solutions describing motion along geodesics.

The energy and the three-momentum of the particle are defined as the components of the *covariant* quadrimomentum:

$$E = \frac{dt}{d\tau}, \quad p_i = a(t)\frac{dx^i}{d\tau} \tag{2.29}$$

τ being the proper time of the particle. The geodesic equation for the 0th component of the 4-velocity gives:

$$\frac{dE}{d\tau} = -H\mathbf{p}^2 \tag{2.30}$$

It follows that the kinetic evolution for the distribution function of free falling particles is:

$$E\frac{\partial f}{\partial t} - H\mathbf{p}^2\frac{\partial f}{\partial E} = 0 \tag{2.31}$$

In the presence of interactions, the right-hand side of this equation is modified by the insertion of an effective collision term which gives the rate of interaction of the particles [4]:

$$E\frac{\partial f}{\partial t} - H\mathbf{p}^2\frac{\partial f}{\partial E} = St(f) \tag{2.32}$$

[4] This is the less rigorous part of the proof, since in a rigorous treatment this term is derived from an integration of the multiparticle distribution functions [37]

Since we are interested only in the total number density of the species, after dividing out by a factor of E we can integrate over the phase space factor to obtain the number density:

$$n(t) = g \int \frac{d^3\mathbf{p}}{(2\pi)^3} f(E,t) \tag{2.33}$$

The second term in the left-hand side of (2.32) can be integrated by parts to obtain the Boltzmann equation:

$$\frac{dn}{dt} + 3Hn = \left(\frac{dn}{dt}\right)_{coll} \tag{2.34}$$

The collisional term for a specific process $\psi + a + b + ... \to c + d + ...$ where ψ is the species under consideration reads:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\int g_\psi \frac{d^3\mathbf{p}_\psi}{(2\pi)^3 2E_\psi} g_a \frac{d^3\mathbf{p}_a}{(2\pi)^3 2E_a} ... g_c \frac{d^3\mathbf{p}_c}{(2\pi)^3 2E_c} ...(2\pi)^4 \delta^4(\mathbf{p}_\psi + \mathbf{p}_a + ... - \mathbf{p}_c - ...) \left[|\mathcal{M}|^2_{\psi+...\to c+...} f_\psi ...(1 \pm f_c)... - |\mathcal{M}|^2_{c+...\to\psi+...} f_c ...(1 \pm f_\psi)...\right] \tag{2.35}$$

where the sign $+$ is for bosons and $-$ for fermions. The statistics blocking or enhancing factors can usually be set to 1 since the distribution functions are generally much smaller than 1 and so can be neglected.

The meaning of the second term in (2.34) becomes evident when we try to compute the rate of change of the total number $N = nV$; since $\frac{dV}{dt} = 3HV$ we find that:

$$\frac{dN}{dt} = V\left(\frac{dn}{dt}\right)_{coll} \tag{2.36}$$

Thus the second term accounts for the dilution of the species due to the expansion of the Universe.

We can simplify the equations absorbing this term by using, instead of n, a quantity which does not change with the expansion; since total entropy S is conserved as well, the ratio $Y = \frac{n}{s}$ will not suffer by the dilution effect. In fact, (2.36) becomes:

$$s\frac{dY}{dt} = \left(\frac{dn}{dt}\right)_{coll} \tag{2.37}$$

It is customary to use as a parameter the temperature of the Universe instead of the cosmic time. Since the epoch of the Universe which we are interested in is the radiation dominated one, the temperature of the thermal bath will vary with time in such a way that the radiation contribution to entropy will be nearly constant; since such contribution depends on temperature as $\sim T^3 a^3$, where the factor of a^3 comes from the total volume of the Universe, it follows that T grows as $\sim \frac{1}{a}$. In the same radiation dominated period, Friedmann equations can be solved explicitly to give:

$$a(t) \sim t^{\frac{1}{2}} \tag{2.38}$$

It follows:

$$t \sim \frac{1}{T^2} \tag{2.39}$$

We can then use as a variable:

$$z = \frac{M}{T} \tag{2.40}$$

where M is any convenient mass scale chosen to adimensionalize z. This implies $t = \alpha z^2$; from (2.38) it then follows:

$$H(z) = \frac{1}{2\alpha z^2} \tag{2.41}$$

We can now substitute into the Boltzmann equation to obtain the following form:

$$s(z)H(z)z\frac{dY}{dz} = \left(\frac{dn}{dt}\right)_{coll} \tag{2.42}$$

The collisional term is typically evaluated by assuming for the momentum part of the distribution function an equilibrium form. Let us give an example of the consequences of such an assumption for the calculation of the term corresponding to a decay [38]; such a term has the form:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\int \frac{d^3p_\psi}{(2\pi)^3 2E_\psi}\frac{d^3p_a}{(2\pi)^3 2E_a}\frac{d^3p_b}{(2\pi)^3 2E_b}(f_\psi(p_\psi) - f_a(p_a)f_b(p_b)) \\ |\mathcal{M}_{\psi\to a+b}|^2(2\pi)^4\delta^4(p_\psi - p_a - p_b) \tag{2.43}$$

where we have set the blocking or enhancing factors to 1 because of our assumption about small occupation numbers.

Upon recognizing in (2.43) the expression for the energy dependent decay rate $\Gamma_D(E_\psi)$, which is related to the decay rate in the COM frame by a Lorentz factor $\Gamma_D(E_\psi) = \Gamma_D\frac{m_\psi}{E_\psi}$, we rewrite it as:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\int \frac{d^3p_\psi}{(2\pi)^3 E_\psi}(f_\psi(p_\psi) - f_a(p_a)f_b(p_b))\Gamma_D m_\psi \tag{2.44}$$

We will assume that, although chemical equilibrium is not maintained (otherwise we would not need the kinetic treatment), kinetic equilibrium is instead maintained through collisions; this implies that the form of the distribution function is $f_\psi(p_\psi) = e^{-\beta E_\psi}\frac{n_\psi}{n_{\psi eq}}$ where $n_{\psi eq}$ is given in (2.18). The assumption of kinetic equilibrium allows us the use of a single temperature for all the species. After using the equality $E_\psi = E_a + E_b$ we find:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\int \frac{d^3p_\psi}{(2\pi)^3 E_\psi}e^{-\beta E_\psi}\Gamma_D m_\psi\left(\frac{n_\psi}{n_{\psi eq}} - \frac{n_a}{n_{aeq}}\frac{n_b}{n_{beq}}\right) \tag{2.45}$$

The integral over E_ψ can now be explicitly done to give:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\gamma_D \left(\frac{n_\psi}{n_{\psi eq}} - \frac{n_a}{n_{aeq}}\frac{n_b}{n_{beq}}\right) \tag{2.46}$$

where:

$$\gamma_D = n_{\psi eq} \frac{K_1\left(\frac{m_\psi}{T}\right)}{K_2\left(\frac{m_\psi}{T}\right)} \Gamma_D \tag{2.47}$$

Analogous calculations can be performed for the case of scattering of two particles into two particles [39]; we obtain the result that, in the process $\psi + c \to a + b$:

$$\left(\frac{dn_\psi}{dt}\right)_{coll} = -\gamma_S \left(\frac{n_\psi}{n_{\psi eq}}\frac{n_c}{n_{ceq}} - \frac{n_a}{n_{aeq}}\frac{n_b}{n_{beq}}\right) \tag{2.48}$$

where:

$$\gamma_S = \frac{T}{32\pi^4} \int_{s_{min}}^{+\infty} ds s^{3/2} \lambda \left(1, \frac{M_\psi^2}{s}, \frac{M_c^2}{s}\right) \sigma(s) K_1 \left(\frac{\sqrt{s}}{T}\right) \tag{2.49}$$

where $s = (p_\psi + p_c)^2$ and $\lambda(a, b, c) = (a - b - c)^2 - 4bc$.

Since in this form of the kinetic equation only ratios of the form $\frac{n}{n_{eq}}$ appear, we can substitute them by the form $\frac{Y}{Y_{eq}}$ in (2.42).

An interesting problem is the deduction of solutions to the Boltzmann equation in the case that the collisional integral vanishes; we will in fact see in the following section that for most of the history of the Universe particles are well described by this very condition.

After the constraint is imposed, (2.32) can be cast into the form:

$$\frac{\partial f(ap, t)}{\partial t} = 0 \tag{2.50}$$

For massless particles, whose energy E coincides with p, a simple solution exists which at any given instant takes the equilibrium form with the energy multiplied by a factor of a; thus, for example, the Bose-Einstein or Fermi-Dirac distribution will become:

$$f(p, t) = \frac{1}{e^{\beta(t_0) p \frac{a(t)}{a(t_0)} - \beta(t_0)\mu} \pm 1} \tag{2.51}$$

which is the same result we would have obtained by taking a standard distribution with a redshifted temperature and chemical potential, both decreasing as $\sim \frac{1}{a}$; another way of finding it would be to note that the entropy density grows as $s \sim T^3$, and, since sa^3 should be constant in an adiabatic expansion, $T \sim \frac{1}{a}$.

For massive particles, in general, no solution exists which depends on the energy alone; in fact, $E = \sqrt{p^2 + m^2}$ cannot be reduced to a function of pa alone. The physical reason for this lies in the fact that a mass introduces a time independent length scale $\sim \frac{1}{m}$[5]; if an equilibrium solution existed, this length would have to increase together with the expansion of the Universe. If $m \gg T$ it is still possible to find an approximate solution by expanding the square root; the Bose-Einstein and Fermi-Dirac distribution will become, for example:

$$f(p,t) = \frac{1}{e^{\beta(t_0)m+\beta(t_0)\frac{p^2a^2(t)}{2ma^2(t_0)}-\beta(t_0)\mu} \pm 1} \tag{2.52}$$

where t_0 is the instant at which free expansion begins. This case is analogous to the previous, but with a temperature which decreases as $\sim \frac{1}{a^2}$.

2.4 Thermal history of the Universe

In the last section we saw that it is possible to parametrize the evolution of the Universe using temperature instead of time; the relationship between the two was schematically derived in the case of a radiation-dominated Universe. This section will be devoted to a qualitative description of the different phases of the early Universe.

We will first discuss the interplay between the two competing effects of expansion, which tends to pull particles out of equilibrium, and interactions, which tend to maintain equilibrium. When the interaction rate is dominant over expansion (that is, $\Gamma \gg H$, Γ being the interaction rate) the Boltzmann equations requires that the collisional terms are nullified by the distribution functions; if this were not true, the time derivative of the distribution function would be extremely high (due to the magnitude of the collisional term) so that any deviation from equilibrium is at this stage nearly immediately cancelled. This implies that at equilibrium particles are described by the equilibrium distributions of Section 2.2 with a temperature which is equal for all species at equilibrium and which depends on time in such a way as to make the total entropy of the Universe constant; because of the large quantity of massless particles present, this generally means at all temperatures of interest that $T \sim \frac{1}{a}$. It is important to note that the distribution function generally differ from those derived at the end of last section since it is forced to follow the evolution of the temperature of the thermal bath, which can be different from its evolution in free expansion. Another fundamental point is that massive particles are suppressed by a factor $\sim e^{-\beta(m-\mu)}$; this implies that, unless they develop a significant chemical potential[6], at equilibrium massive particles are negligible.

When the expansion rate is dominant ($\Gamma \ll H$) the collision term is negligible; in this case the solutions become those derived at the end of last section. The evolution of each species, therefore, is characterized by two fundamental moments; the decoupling time, at which $H \sim \Gamma$, and the time at which it becomes non relativistic, when $T \sim M$, M being the mass of the particle.

[5]In a quantum context this length coincides with the minimum uncertainty on the particle position, that is, its localization length; under this scale pair production makes it impossible to localize the particle.

[6]This can typically happen when they are protected by a particular symmetry.

If the particle becomes massive ($T \sim M$) before decoupling, then, as we have seen, its concentration at the time of decoupling will be negligible unless it has developed a chemical potential which protects it; this is, for example, the case of baryons, whose total number at sufficiently low temperature ($T \ll 300\ GeV$) must be conserved because of the absence of baryon number violating processes. If decoupling happens at a time at which $T \sim M$, it is possible for the species to become "freezed out" at an equilibrium value of na^3.

We willl now briefly sketch the most important events in the thermal history of the Universe [40]:

- at a temperature $T \sim 100\ GeV$ the electroweak phase transition breaks the symmetry group to the standard $SU(3) \times SU(2) \times U(1)$ by giving an expectation value to the Higgs; at this temperature all particles are massless and coupled to the thermal bath. At this time there is already a non vanishing baryon number density n_B, which, because of conservation of baryon number[7], decreases exactly as $n_B \sim \frac{1}{a^3}$;

- at $T \sim 150\ MeV$ quark confinement appears; before this moment quarks and gluons were free, while at this temperature they become confined into bound states of hadrons;

- at $T \sim 1\ MeV$ neutrinos are the first to decouple from equilibrium; in the instantaneous decoupling approximation, they simply start a free expansion with $T_\nu \sim \frac{1}{a^3}$ and with an initial temperature equal to that of the thermal bath;

- at $T \sim 500\ keV$ electrons and positron annihilate and become non relativistic; this is the main reason for neutrino temperature being different from that of the thermal bath. In fact, were it not for $e^+ - e^-$ annihilation, both temperatures would decrease with the same law; instead, pair annihilation into photons causes a "heating" of the photon temperature which in this phase does not decrease exactly as $\sim \frac{1}{a}$;

- at $T \sim 100\ keV$ nucleons have decoupled from the thermal bath and start the chain of nuclear reactions leading to Big Bang Nucleosynthesis; after primordial nucleosynthesis has happened, about 75% of baryonic matter is in the form of protons, 25% in the form of ^{4}He and a negligible part composing heavier nuclei;

- at $T \sim 0,75\ eV$ $\Omega_{matter} = \Omega_{rad}$; from now on matter will dominate over radiation;

- at $T \sim 0,25\ eV$ hydrogen atoms form (this goes under the name of *recombination*); matter is now nearly completely in the form of neutral atoms, so that photons decouple, starting to form the Cosmic Microwave Background.

[7] In the absence of sphaleron and GUT interactions.

The present temperature is about $T \sim 0,24\ meV$ and the Universe is dominated by dark energy, as evidenced by the accelerated expansion revealed from analysis of Type Ia Supernovae.

The processes we will be mostly interested in all happen at a temperature above 100 GeV, at the time of the formation of a non vanishing baryon number. Because of the behavior $n_B \sim \frac{1}{a^3}$, we can say that, apart from a correction due to $e^+ - e^-$ annihilation, the ratio $\eta = \frac{n_B}{n_\gamma}$ (n_γ being the photon number density) is nearly constant during all the subsequent evolution.

One might suppose that at temperatures above 100 GeV all Standard Model interactions are always at equilibrium; this is actually not true, due to the fact that the species are massless. As a consequence, while the Hubble constant grows as $\sim T^2$, the decay rate by dimensional analysis has to grow as $\sim T$; therefore there will be a maximum temperature above which the reactions are out of equilibrium. We will be mostly interested in the regime $10^8\ GeV \leq T \leq 10^{12}\ GeV$; the reactions which are already at equilibrium at this time will be detailed in a later section.

2.5 Baryogenesis

A major problem in cosmology is the explanation of the cosmic asymmetry between matter and antimatter. On a cosmic scale no macroscopic concentrations of antimatter are expected; in fact, boundaries between regions composed of matter and antimatter would produce gamma ray emissions which are not observed.

On the other hand, a macroscopic concentration of baryonic matter can be measured; we can quantify it using various parameters. The most common choices are $\eta = \frac{n_B}{n_\gamma}$ and $Y_B = \frac{n_B}{s}$; an equivalent parametrization is the cosmological parameter Ω_B. The value of η can be inferred in two independent ways [40].

On one hand, it is possible to use η as an input parameter for the kinetic equations of the Big Bang Nucleosynthesis to predict the concentrations of light elements (D, ^3He, ^4He and ^7Li). On the other hand, baryon concentration plays a fundamental role in determining the acoustic peaks in CMB; in fact, at the time of recombination, when CMB was produced, the Universe was basically made of a plasma of photons and baryonic matter, so acoustic hydrodynamic oscillations depended on baryon concentration.

Both these arguments lead to the following values for baryon concentration:

$$\eta = \frac{n_B}{n_\gamma}|_0 = (6.21 \pm 0.16) \times 10^{-10} \tag{2.53}$$

$$Y_B = \frac{n_B}{s}|_0 = (8.75 \pm 0.23) \times 10^{-11} \tag{2.54}$$

where the subscript 0 indicates that the values refer to present time (as was mentioned in the previous section, only a small correction due to $e^+ - e^-$ annihilation is expected for the present time value).

By the previous arguments we know that the concentrations of antibaryons is completely negligible with respect to the concentration of baryons, so that we may as well interpret η as $\frac{n_B - n_{\overline{B}}}{n_\gamma}$. The question arises as to whether we should consider η as an initial condition on the Universe itself or we should look for a way of calculating it from first principles. Apart from aesthetical advantages, there is an important theoretical reason for trying to deduce η from first principles. In fact, inflation would have washed out any preexisting baryon asymmetry; it follows that such an asymmetry, if inflation has happened, has to be regarded as dynamically generated.

The possibility of describing the spontaneous generation of a non vanishing baryonic number from a state characterized by a net zero baryonic number requires some specific conditions on the microphysics at the basis of our cosmology; these conditions were identified for the first time by Sakharov [1]. They can be summarized as follows:

1. There should exist baryon number violating interactions. This condition is obvious in that we want to pass from a state with zero baryonic number to a state with a non zero baryonic number. As has been outlined in Chapter 1, the Standard Model naturally contains baryon number violating interactions of topological nature, which appear in the form of instantons and sphalerons; as it was noted there, although at zero temperature their effects are strongly suppressed, thermal fluctuations can make their rates appreciable. The vanishing of the anomaly in the $B-L$ current gives a selection rule for these processes that $\Delta B = \Delta L = 3n$, where the factor of 3 comes from the number of families. We mention here that in GUT baryon and lepton number violating interactions already appear at tree level; they are in fact suppressed by a factor $\sim \frac{1}{M}$, where M is the breaking scale of the gauge unification group.[8]

2. There should exist C and CP violating interactions. In fact, if this were not true, then the number of baryons and antibaryons would remain equal for all the time as it was at the beginning. The Standard Model contains C and CP violating interactions, due to the phase in the CKM matrix; we also remark that, by adding mass terms to neutrinos, a new source of CP violation comes from the phase of the PMNS matrix.

3. The relevant processes for baryogenesis should happen out of equilibrium. This can be understood heuristically by noting that, by definition, the equilibrium state is macroscopically invariant under time reversal; since the fundamental theory is assumed to be CPT invariant, such a state also preserves CP, which means that no baryon number can arise. More quantitatively, at chemical equilibrium one can impose the relevant constraints on the chemical potentials which admit as a solution the vanishing of $\mu_B = \mu_{\overline{B}} = 0$ and thus a vanishing baryon number. The Standard Cosmological Model naturally contains an out of equilibrium period during

[8]This typically means a scale $\sim 10^{15}\ GeV$, which is above the range of interest for leptogenesis.

the electroweak phase transition; however, it has been proven that the experimental lower bound on the Higgs mass is such that the transition is not a first order one[9] [41].

Many possibilities have been proposed which met all of the above condition. In the context of Standard Model alone, CP violation is too small to produce consistent baryogenesis; furthermore, as we noted above, the electroweak phase transition is of second order, and it is not able to reproduce a succesful baryogenesis. It is necessary to extend the Standard Model to include CP and baryon number violating processes. A set of models which are collectively known as electroweak baryogenesis are based on a suitable modification of the Higgs content of the theory which makes the electroweak transition a second order one.

An alternative possibility is to introduce CP and *lepton* number violating processes in order to produce a non vanishing lepton number, which is then converted, by means of sphalerons, to baryon number; this scenario goes under the name of *baryogenesis via leptogenesis* [2]. A particularly attractive possibility lies in the identification of the above mentioned process with the seesaw mechanism; in fact, as was described in Chapter 1, the heavy right-handed neutrino possesses an interaction vertex with the Higgs and the left-handed neutrino which causes it to decay through a generally CP non invariant amplitude. The out of equilibrium processes would then be provided by the freezing out of right-handed neutrinos, whose Yukawa couplings are supposed to be such that at the temperature of leptogenesis they are just decoupling.

When the seesaw mechanism is identified with the process responsible for leptogenesis the model is called *thermal leptogenesis.*

2.6 Thermal leptogenesis

In the thermal leptogenesis scenario it is supposed that the asymmetry is generated by the CP violating decay of heavy neutrinos; the mechanism is inspired by the delayed decay model, originally proposed for B and CP violating decays in [42].

The basic requirement is that the heavy decaying species runs out of equilibrium at a temperature $T \leq M$, where M is the mass of the decaying particle. In fact, suppose this was not true; then, at the time at which the species freezes out, Boltzmann equation for the decaying species still admits an equilibrium form for the particle distribution with a temperature $T \sim \frac{1}{a}$. This result was proved in the above section; there we found it to be rigorously true for species whose mass is negligible at the temperature of interest. We can now explain the physical reason behind this requirement; the point is that, if the mass is of the same order of magnitude or bigger than temperature, the concentration of heavy particles is suppressed by a factor $\sim e^{-\frac{M}{T}}$. In order for such a suppression to

[9] A heuristic understanding of the reason why there is a maximum Higgs mass which leads to a first order phase transition comes from the observation that for high masses the theory substantially behaves as if it was a simple ϕ^4 theory without any other component; as is well known, for such a theory the phase transition is of second order

appear, decays have to happen more frequently than inverse decays; this is the out of equilibrium condition we were looking for, which permits the production of a non vanishing lepton number.

It might be objected that, although in the freezing out of the massive species an asymmetry is produced because of the higher rate of decays over inverse decays, the very production of the species had to happen in an asymmetric way which should make the total baryon asymmetry vanish; the response to the objection lies in the so called *washout* processes. In fact, decays, inverse decays and scattering processes wash out any asymmetry which is preexisting to the freezing out, including the "anti-asymmetry" which was generated in the thermal production of right-handed neutrinos.

The condition $T \sim M$ requires thermal leptogenesis to happen at a temperature of the order of the masses of right-handed neutrinos; a lower bound [43] known as the Davidson-Ibarra (DI) bound, which is detailed a little more in Appendix 2, can be put on such masses of order 10^8 GeV for successful leptogenesis. We underline that this bound is actually derived under the condition that only a single neutrino species contributes; if two neutrinos are degenerate in mass leptogenesis can be resonantly enhanced and the DI bound can be evaded; as will be seen in detail in Chapter 3, however, we are interested in a regime which is not fully resonant. The conclusion which can be drawn is that leptogenesis has to happen at a temperature higher than 10^8 GeV.

The quantitative description of the proposed mechanism requires an analysis of the Lagrangian involving the right-handed and left-handed neutrinos [39]. The relevant terms are, apart from kinetic terms:

$$\mathcal{L} = -\frac{1}{2}\overline{N}_i N_i^c M_i - \lambda_{\alpha i}\bar{l}_\alpha N_i \phi^c - h.c. \tag{2.55}$$

We have chosen right-handed neutrinos to be mass eigenstates; we remind the reader that they are Majorana particles carrying zero lepton number. By l_α we describe the left-handed lepton doublet; at the temperatures of interest, the Higgs field still has vanishing expectation value.

At tree level neutrino decay happens with an amplitude $\mathcal{M} \sim \lambda_{\alpha i}$; since the rate depends on the squared modulus of this matrix element it is evident that there is no CP violation at tree level and we must include one loop amplitudes; the contributing diagrams are shown in Figure 2.1. In the squared modulus of the amplitudes there will now appear CP violating interference terms of order $\mathcal{O}(\lambda^4)$.

Let us define the decay rates $\Gamma_{N_i \to l_\alpha + \phi}$, $\Gamma_{N_i \to \bar{l}_\alpha + \overline{\phi}}$ and the inverse decay rates $\Gamma_{l_\alpha + \phi \to N_i} = \Gamma_{N_i \to \bar{l}_\alpha + \overline{\phi}}$ and $\Gamma_{\bar{l}_\alpha + \overline{\phi} \to N_i} = \Gamma_{N_i \to l_\alpha + \phi}$ because of CPT invariance. We also introduce the CP asymmetry defined as:

$$\epsilon_{i\alpha} = \frac{\Gamma_{N_i \to l_\alpha + \phi} - \Gamma_{N_i \to \bar{l}_\alpha + \overline{\phi}}}{\Gamma_{N_i \to l_\alpha + \phi} + \Gamma_{N_i \to \bar{l}_\alpha + \overline{\phi}}} \tag{2.56}$$

The Boltzmann equation for right-handed neutrino takes the standard form with

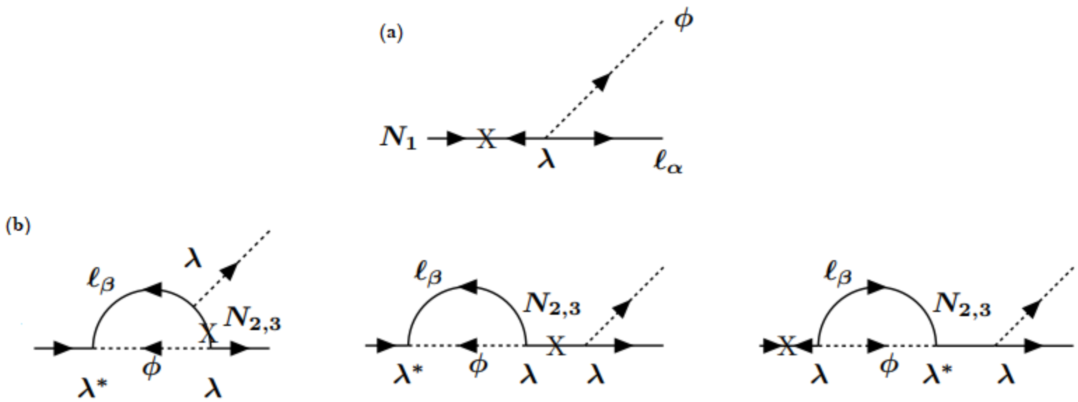


Figure 2.1: Diagrams contributing to order $\mathcal{O}(\lambda^2)$ in the amplitude to right-handed neutrino decay (a) at tree level and (b) at one loop level (adapted from [39]).

the collisional integral being determined by decays and inverse decays:

$$sHz\frac{dY_i}{dz} = -\sum_\alpha \gamma_{N_i\to l_\alpha+\phi}\left(\frac{Y_i}{Y_{ieq}} - \frac{Y_\alpha}{Y_{\alpha eq}}\frac{Y_\phi}{Y_{\phi eq}}\right) - \sum_\alpha \gamma_{N_i\to\bar{l}_\alpha+\overline{\phi}}\left(\frac{Y_i}{Y_{ieq}} - \frac{Y_{\overline{\alpha}}}{Y_{\alpha eq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}}\right) \tag{2.57}$$

In (2.57) Y_i, Y_α, $Y_{\overline{\alpha}}$, Y_ϕ, $Y_{\overline{\phi}}$ are respectively the yields of the ith right-handed neutrino, the αth lepton and antilepton, the Higgs and the antiparticle of the Higgs (of course $Y_{\alpha eq} = Y_{\overline{\alpha}eq}$ and $Y_\phi = Y_{\overline{\phi}}$).

At the temperatures of interest, set by the DI bound, all Standard Model interactions are at equilibrium; in particular, this implies that the Higgs (and its antiparticle) concentration can be set to its equilibrium value. The same reasoning can be applied to lepton concentrations. There is, however, a slight complication; in fact, if we blindly set Y_α to its equilibrium value, we do not obtain any leptogenesis. The subtlety lies in the fact that we are allowed to set Y_α and $Y_{\overline{\alpha}}$ separately to their equilibrium values apart from small corrections which become important only when we pass to their difference $\mathcal{L}_\alpha = Y_\alpha - Y_{\overline{\alpha}}$. Since the Higgs is in chemical equilibrium with all other species, its asymmetry $\Phi = Y_\phi - Y_{\overline{\phi}}$ will have to be taken into account too.

Upon introducing the total neutrino decay rates and its thermal average $\Gamma_i = \sum_\alpha \left(\Gamma_{N_i\to l_\alpha+\phi} + \Gamma_{N_i\to\bar{l}_\alpha+\overline{\phi}}\right)$ and $\gamma_i = \sum_\alpha \left(\gamma_{N_i\to l_\alpha+\phi} + \gamma_{N_i\to\bar{l}_\alpha+\overline{\phi}}\right)$ Boltzmann equation becomes:

$$sHz\frac{dY_i}{dz} = -\gamma_i\left(\frac{Y_i}{Y_{ieq}} - 1\right) \tag{2.58}$$

The other equation of interest is an evolution equation for $\mathcal{L}_\alpha$; before deriving it, let us introduce the useful notation:

$$\Gamma_{i\alpha} = \Gamma_{N_i \to l_\alpha + \phi} + \Gamma_{l_\alpha + \phi \to N_i} \tag{2.59}$$

and similar notations for the thermally averaged rate $\gamma_{i\alpha}$.

The collisional part of the integral receives contribution to order $\mathcal{O}(\lambda^4)$ both from tree and loop level amplitudes of decays and inverse decays and from tree level amplitudes of scattering processes; a further contribution comes from sphaleronic processes:

$$sHz\frac{dY_\alpha}{dz} = \left(\frac{dn_\alpha}{dt}\right)_{dec} + \left(\frac{dn_\alpha}{dt}\right)_{scat} + \left(\frac{dn_\alpha}{dt}\right)_{sphal} \tag{2.60}$$

It will be later seen that the sphaleronic contribution needs not be computed; for the following we will avoid writing it in the equations, although it has to be kept in mind that its contribution is fundamental, and only later take it into account.

We will be finally interested in the difference between this equation and the corresponding one for $Y_{\overline{\alpha}}$; it might then be thought that the scattering rate, being evaluated from the squared modulus of tree level amplitudes, will not produce any source term for the asymmetry. This turns out not to be true because of a subtlety; in fact, the scattering processes $l_\alpha + \phi \to l_\alpha + \phi$ can be mediated by on-shell right-handed neutrinos. Such processes have already been taken into account from decays and inverse decays terms, and are therefore to be subtracted from the scattering amplitudes. This is done most easily by using a subtracted thermal rate; a typical example would be:

$$\gamma^{(s)}_{l_\alpha + \phi \to l_\beta + \phi} = \gamma_{l_\alpha + \phi \to l_\beta + \phi} - \gamma_{l_\alpha + \phi \to N_i}\frac{1 + \epsilon_{i\beta}}{2} \tag{2.61}$$

where the subtraction rate has been determined from multiplying the decay rate for the process $l_\alpha + \phi \to N_i$ by the branching ratio of the unstable neutrino decay into l_β and ϕ.

The calculation for 3 flavors is complicated by the presence of many indices; we will now do a conceptually analogous computation for a single flavour case, which can be straightforwardly generalized to the flavoured case. We stress, however, that this must be regarded as a purely pedagogical exercises, since, with a single flavor, there is no CP asymmetry (see Appendix A for more details).

We will denote by Y_N the right-handed neutrino yield, Y_L, $Y_{\overline{L}}$ and $\mathcal{L} = Y_L - Y_{\overline{L}}$ the lepton and antilepton yields and asymmetry respectively and by Y_ϕ, $Y_{\overline{\phi}}$ and Φ the Higgs and anti Higgs yields and its asymmetry; thermal decay rates will be denoted by:

$$\gamma_{N \to L + \phi} = \gamma_{\overline{L} + \overline{\phi} \to N} = \frac{\gamma}{2}(1 + \epsilon), \quad \gamma_{N \to \overline{L} + \overline{\phi}} = \gamma_{L + \phi \to N} = \frac{\gamma}{2}(1 - \epsilon) \tag{2.62}$$

To this order in λ the unsubtracted scattering rates are equal for all processes $L + \phi \to L + \phi$, $L + \phi \to \overline{L} + \overline{\phi}$, $\overline{L} + \overline{\phi} \to L + \phi$, $\overline{L} + \overline{\phi} \to \overline{L} + \overline{\phi}$ and will be denoted by γ_s. The equations for Y_L and $Y_{\overline{L}}$, taking into account the previous subtraction terms, are:

$$
\begin{aligned}
sHz\frac{dY_L}{dz} &= -\frac{\gamma}{2}(1-\epsilon)\frac{Y_L}{Y_{Leq}}\frac{Y_\phi}{Y_{\phi eq}} + \frac{\gamma}{2}(1+\epsilon)\frac{Y_N}{Y_{Neq}} - \\
&\left(\gamma_s - \gamma(1-\epsilon)\frac{1+\epsilon}{4}\right)\frac{Y_L}{Y_{Leq}}\frac{Y_\phi}{Y_{\phi eq}} + \left(\gamma_s - \gamma(1+\epsilon)\frac{1-\epsilon}{4}\right)\frac{Y_L}{Y_{Leq}}\frac{Y_\phi}{Y_{\phi eq}} - \\
&\left(\gamma_s - \gamma(1-\epsilon)\frac{1-\epsilon}{4}\right)\frac{Y_L}{Y_{Leq}}\frac{Y_\phi}{Y_{\phi eq}} + \left(\gamma_s - \gamma(1+\epsilon)\frac{1+\epsilon}{4}\right)\frac{Y_{\overline{L}}}{Y_{Leq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}}
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
sHz\frac{dY_{\overline{L}}}{dz} &= -\frac{\gamma}{2}(1+\epsilon)\frac{Y_{\overline{L}}}{Y_{Leq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}} + \frac{\gamma}{2}(1-\epsilon)\frac{Y_N}{Y_{Neq}} - \\
&\left(\gamma_s - \gamma(1-\epsilon)\frac{1+\epsilon}{4}\right)\frac{Y_{\overline{L}}}{Y_{Leq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}} + \left(\gamma_s - \gamma(1+\epsilon)\frac{1-\epsilon}{4}\right)\frac{Y_{\overline{L}}}{Y_{Leq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}} - \\
&\left(\gamma_s - \gamma(1+\epsilon)\frac{1+\epsilon}{4}\right)\frac{Y_{\overline{L}}}{Y_{Leq}}\frac{Y_{\overline{\phi}}}{Y_{\phi eq}} + \left(\gamma_s - \gamma(1-\epsilon)\frac{1-\epsilon}{4}\right)\frac{Y_L}{Y_{Leq}}\frac{Y_\phi}{Y_{\phi eq}}
\end{aligned}
\tag{2.64}
$$

By subtracting them from one another and keeping only terms which are of first order in ϵ (we remind that $\mathcal{L}$ and Φ are of the same order as ϵ) we find the equation for the asymmetry:

$$
sHz\frac{d\mathcal{L}}{dz} = \gamma\epsilon\left(\frac{Y_N}{Y_{Neq}} - 1\right) - 2\gamma_s\left(\frac{\mathcal{L}}{Y_{Leq}} + \frac{\Phi}{Y_{\phi eq}}\right) \tag{2.65}
$$

A simple check on our result is the absence of a source term for the asymmetry when right-handed neutrinos are at equilibrium: had we not included the subtraction term in the scattering rate, we would have obtained the unphysical result that, even under equilibrium conditions, an asymmetry is produced.

The last term in (2.65) is called the *washout* term, since it has the effect of destroying any asymmetry which precedes the out of equilibrium phase of right-handed neutrinos. We notice that $L + \phi \to L + \phi$ scattering is dominated by on shell right-handed neutrino mediated collisions, so that we can write:

$$
\gamma_s \sim \frac{\gamma}{4} \tag{2.66}
$$

neglecting terms of first order in ϵ.

Before generalizing the formulas obtained to the case of 3 flavours, let us examine the assumptions made in the previous calculation. We have neglected all processes involving Standard Model particles other than leptons; a part of these processes will involve right-handed neutrinos (e.g., $N \to q_r + q_L + l_\alpha$), while others will include only leptons and Higgs particles (typical examples would be $\phi + \phi \to l_\alpha + l_\beta$). An accurate analysis confirms the intuitive results that, to first order in ϵ, the former modify the total decay rate of right-handed neutrinos while the latter influence the washout terms without modifying the structure of

Boltzmann equations; their corrections are generally small and will be neglected in the following.

Upon reintroducing flavor indices we find (using the notation of [44]:

$$sHz\frac{d\mathcal{L}_\alpha}{dz} = \sum_i \epsilon_{i\alpha}\gamma_i \left(\frac{Y_i}{Y_{ieq}} - 1\right) - \sum_i \frac{\gamma_{i\alpha}}{2}\left(\frac{\mathcal{L}_\alpha}{Y_{\alpha eq}} + \frac{\Phi}{Y_{\phi eq}}\right) \tag{2.67}$$

where $\gamma_{i\alpha}$ is the total thermally averaged decay rate of right-handed neutrinos into the αth species (both particle and antiparticle) and (2.66) has been used. We will use the notation:

$$\gamma_{i\alpha} = \gamma_i P_{i\alpha} \tag{2.68}$$

We recall that Y_{ieq} is the equilibrium yield for a massive particle with 2 degree of freedom (since it is a Majorana particle) $Y_{ieq} = \frac{45M_i^2z^2}{2\pi^4 g_S^* M^2}K_2(\frac{M_i z}{M})$; $Y_{\alpha eq}$ is the equilibrium yield for a fermionic relativistic particle with 2 degrees of freedom (since it describes a doublet of particles) $Y_{\alpha eq} = \frac{135\zeta(3)}{4\pi^4 g_S^*}$; $Y_{\phi eq}$ is the equilibrium yield for a bosonic relativistic particle with 2 degrees of freedom $Y_{\phi eq} = \frac{45\zeta(3)}{g_S^*\pi^4}$.

Decay rates and asymmetries are given by the following formulas:

$$\Gamma_i = \frac{M_i}{8\pi}(\lambda^\dagger\lambda)_{ii} \tag{2.69}$$

$$P_{i\alpha} = \frac{\lambda^\dagger_{i\alpha}\lambda_{\alpha i}}{(\lambda^\dagger\lambda)_{ii}} \tag{2.70}$$

$$\epsilon_{i\alpha} = \frac{1}{8\pi}\sum_{k\neq i}\frac{\mathrm{Im}\left\{\lambda^\dagger_{i\alpha}\lambda_{\alpha k}\left(\lambda^\dagger\lambda\right)_{ik}\right\}}{(\lambda^\dagger\lambda)_{ii}} f\left(\frac{M_k^2}{M_i^2}\right) + \frac{1}{8\pi}\sum_{k\neq i}\frac{\mathrm{Im}\left\{\lambda^\dagger_{i\alpha}\lambda_{\alpha k}\left(\lambda^\dagger\lambda\right)_{ki}\right\}}{(\lambda^\dagger\lambda)_{ii}} g\left(\frac{M_k^2}{M_i^2}\right) \tag{2.71}$$

where f and g are loop functions defined by:

$$f(x) = \sqrt{x}\left[\frac{1-x}{(1-x)^2 + \left(\frac{\Gamma_i}{M_i} - x\frac{\Gamma_k}{M_k}\right)^2} + 1 - (1+x)\log\frac{1+x}{x}\right] \tag{2.72}$$

$$g(x) = \frac{1-x}{(1-x)^2 + \left(\frac{\Gamma_i}{M_i} - x\frac{\Gamma_k}{M_k}\right)^2} \tag{2.73}$$

As was noted above we actually neglected a fundamental point, which makes our derivation invalid; this is the presence of sphaleron reactions, which should be added to the collisional integral. Fortunately, there is an easy way to get rid of the sphaleronic term by noting that Standard Model conserves $B - L$; this means that, if we denote by $Y_{\Delta\alpha} = \frac{Y_{\Delta B}}{3} - Y_{\Delta L_\alpha}$, its collisional integral will not contain any sphaleronic contribution (notice that $Y_{\Delta L_\alpha} = \mathcal{L}_\alpha + Y_{\Delta l^R_\alpha}$ contains the contribution of the asymmetries both of the left-handed and the right-handed Standard Model leptons). Since, at the temperatures of interest, baryonic number is conserved apart from topological processes, we can directly write:

$$sHz\frac{dY_{\Delta\alpha}}{dz} = -\sum_i \epsilon_{i\alpha}\gamma_i\left(\frac{Y_i}{Y_{ieq}} - 1\right) + \sum_i \frac{\gamma_{i\alpha}}{2}\left(\frac{\mathcal{L}_\alpha}{Y_{\alpha eq}} + \frac{\Phi}{Y_{\phi eq}}\right) \tag{2.74}$$

The final point in our analysis is the derivation of a connection between the variables $\mathcal{L}_\alpha$ and Φ and $Y_{\Delta\alpha}$; such a connection derives from the equilibrium conditions of all Standard Model reactions. As is well known, in fact, for an equilibrium reaction the sum of the chemical potentials of all particles (taken with the positive sign for reagents and with the negative sign for products) vanishes.

This is a particularly subtle part of the study in that it involves the determination of which reactions are at equilibrium in the particular temperature regime we are interested in; since leptogenesis happens at $T \sim M$, this regime will be fixed by the right-handed neutrino masses. It will be seen in Chapter 3 that the typical masses range from 10^9 to 10^{11} GeV; here the following reactions are at equilibrium [39]-[45].

- The Yukawa couplings with the Higgs are at equilibrium for all species apart from e, u and d; these imposes 6 constraints (for t,b,c,s,τ and μ) on the chemical potentials:

$$\begin{aligned} \mu_t = \mu_{q_3} + \mu_\phi, \quad \mu_b = \mu_{q_3} - \mu_\phi, \quad \mu_c = \mu_{q_2} + \mu_\phi \\ \mu_s = \mu_{q_2} - \mu_\phi, \quad \mu_\tau = \mu_{l_\tau} - \mu_\phi, \quad \mu_\mu = \mu_{l_\mu} - \mu_\phi \end{aligned} \tag{2.75}$$

- The electroweak sphaleron reactions are at equilibrium; this imposes the further constraint on the chemical potential of left-handed quark and lepton doublets $3\sum_\alpha \mu_{q_\alpha} + \sum_\alpha \mu_{l_\alpha} = 0$.

- An equilibrium hypercharge neutrality condition has to be imposed on all species of the form $\sum_\alpha \mu_{q_\alpha} + 2\sum_\alpha \mu_{u_\alpha} - \sum_\alpha \mu_{d_\alpha} - \sum_\alpha \mu_{l_\alpha} - \sum_\alpha \mu_{e_\alpha} + 2\mu_\phi = 0$

- A QCD sphaleron reaction is also at equilibrium at this temperature, leading to the condition $2\sum_\alpha \mu_{q_\alpha} - \sum_\alpha \mu_{u_\alpha} - \sum_\alpha \mu_{d_\alpha} = 0$.

- The three baryon flavour asymmetries are all equal; this happens because they are equal as an initial condition, and, as soon as flavour-changing interactions come into equilibrium, they drive the asymmetries to equality. We can therefore impose another condition $2\mu_{q_1} + \mu_u + \mu_d = 2\mu_{q_2} + \mu_c + \mu_s = 2\mu_{q_3} + \mu_t + \mu_b$.

- Since Yukawa couplings for e,u and d have not come into equilibrium, we can assume their asymmetries to be conserved and equal to their initial value, which is presumably 0; this imposes the further constraints $\mu_e = 0$ and $\mu_u = \mu_d$.

The previous constraints have been analytically solved to express the quantities of interest in terms of the $\mu_{\Delta\alpha}$ defined above:

$$\mathcal{L}_\alpha = A_{\alpha\beta} Y_{\Delta\beta}, \quad \Phi = C_\beta Y_{\Delta\beta} \tag{2.76}$$

where:

$$A = \frac{1}{2148} \begin{pmatrix} -906 & 120 & 120 \\ 75 & -688 & 28 \\ 75 & 28 & -688 \end{pmatrix}, \quad C = -\frac{1}{358} \begin{pmatrix} 37 \\ 52 \\ 52 \end{pmatrix} \tag{2.77}$$

A somewhat tricky factor of 2 has to be included in the coefficients for the Higgs boson because of the difference in (2.25) and (2.26) for bosons and fermions.

After these conditions are included, (2.58) and (2.74) form a system of 6 differential equations in the six unknown functions Y_i and $Y_{\Delta\alpha}$ which can be numerically solved to deduce the asymptotic values. From these we can find the baryon asymmetry which has been produced through a relation deriving from the same chemical equilibrium conditions as the others; these conditions, however, must not be imposed at the time of leptogenesis. In fact, after leptogenesis has happened, $Y_{\Delta\alpha}$ becomes a conserved quantity while baryon number keeps changing due to sphaleronic interactions. The final value of baryon number yield must be evaluated, therefore, at the moment the sphaleron runs out of equilibrium; this happens at a lower temperature, when e, u and d Yukawa coupling have also come to equilibrium. A straightforward calculation gives the standard result:

$$Y_{\Delta B} = \frac{28}{79} \sum_\alpha Y_{\Delta\alpha} \tag{2.78}$$

This has been deduced in the case of a single Higgs multiplet; in our specific model we will need the extension to the case of two Higgs multiplets, whereby the factor $\frac{28}{79}$ is changed to $\frac{8}{23}$.

We reserve some final comments on the assumption at the basis of the use of the Boltzmann formalism. This is derived in the context of a semiclassical approximation, as was discussed before. It is possible, however, to obtain the correct equations by writing the Schwinger-Dyson equations for the density operator and perturbatively solve them; the terms in the perturbation series quickly become involved and even the use of the diagrammatic technique in the closed time path formalism [46] leads to long calculations. The results [47] differ from the semiclassical ones used here in that they show a typical memory effect in which the concentrations depend on the data from intermediate times on time scales of order $\sim \frac{1}{E}$, where E is the typical energy of the particles involved. No real estimate exist in literature of the importance of such effect.

Chapter 3

Leptogenesis in an $SO(10)$ inspired context

3.1 Fermion masses in $SO(10)$

In this section we will introduce the notation and the model in the context of which we have studied the baryogenesis problem. In Chapter 2 we found that a kinetic analysis requires as input data the Yukawa couplings $\lambda_{\alpha i}$ of right-handed neutrinos with the Higgs boson and the left-handed leptons and the right-handed neutrino masses M_i; the former can be equivalently replaced by the Dirac neutrino mass matrix $M_{D\alpha i}$ since, upon spontaneous symmetry breaking, they differ by a factor of v, the vacuum expectation value of the Higgs. Such a mass matrix can be expected to be quite hierarchical, similarly to the Yukawa coupling of the heavy leptons with the Higgs. We have focused in our analysis on an $SO(10)$ inspired mass relation which will be analyzed in detail below; however, we remark that this is just an example of a possible hierarchical structure for the Dirac mass matrix, for which $SO(10)$ plays the role of an inspiration. We have not deeply investigated all the consequences of the specific $SO(10)$ model which has been used as a basis; this might be the object of a further analysis.

In Chapter 1 we deduced a number of mass relations which hold in the case that the representation of $SO(10)$ used to give mass to the leptons is a **10** or a **126**[1]; as was described there, in both cases these relations induce a proportionality between the Dirac and the quark mass matrices with coefficients of order unity. Thus we could write them in the form:

$$M_u \sim M_{D\nu} \tag{3.1}$$

(In the following we will neglect the subscript ν). In our work, we have

[1] We recall that the **126**, at a renormalizable level, is necessary in order to allow for the right-handed neutrino Majorana mass term which will generate the seesaw mechanism

assumed that this proportionality was a simple equality, as if only the **10** were present:

$$M_u = M_D \tag{3.2}$$

Let us write the mass terms in the Lagrangian:

$$-\frac{1}{2} M_{Rij} \overline{N}_i N_j^c - M_{D\alpha i} \overline{\nu}_\alpha N_i - h.c. \tag{3.3}$$

Differently from what we did in our treatment of thermal leptogenesis in Chapter 2, we have here used a basis for right-handed neutrinos which is different from their mass eigenstates; we cannot assume, in fact, that the "natural" basis for $SO(10)$, in the context of which (3.2) holds, coincides with the mass eigenstates basis. In (3.3) Latin indices refer to right-handed neutrinos and Greek indices refer to Standard Model flavours.

If we make the additional assumption that heavy leptons (electron, muon and tauon) mass eigenstates coincide with weak interaction eigenstates, then it follows that the matrix M_D is diagonalized by the same CKM matrix which appears in quarks weak interaction mixing. This specific assumption would not be admissible without further analysis in a real $SO(10)$ model; in fact, it has been shown [48] that the best fit parameters are obtained for a matrix which is not diagonal. Since we are interested in $SO(10)$ mass relations only as an example, however, we have not further investigated this point.

The matrix M_D is generically not symmetrical; however, in a minimal approach to our simple $SO(10)$ model we can imagine symmetry to be broken by a **10** and a **126**, which, as we saw before, possess a symmetrical Yukawa coupling matrix. This leads to the assumption that M_D is symmetrical; a natural implication is that it can be diagonalized through a single unitary matrix:

$$M_D = V_L^\dagger M_D^{diag} V_L^* \tag{3.4}$$

The seesaw mechanism now leads to a neutrino mass matrix of the form:

$$M_\nu = -M_D M_R^{-1} M_D^T \tag{3.5}$$

We can invert this relation to express M_R in terms of M_ν and M_D, which, by our $SO(10)$ hypothesis, is equal to M_u:

$$M_R = -M_D M_\nu^{-1} M_D \tag{3.6}$$

We can easily pass to the representation where M_R is diagonal by noting that M_R is a product of symmetrical matrices, and thus is symmetric; this means that we can write:

$$M_R = W M_R^{diag} W^T \tag{3.7}$$

In the basis of right-handed neutrino eigenstates defined by (3.7) the Dirac mass matrix becomes:

$$\hat{M}_D = M_D W^* \tag{3.8}$$

Since this was the basis in which all of our derivations about thermal leptogenesis were made, the λ matrix introduced in the previous Chapter is linked to the Dirac mass matrix by:

$$\hat{M}_D = \lambda v \tag{3.9}$$

3.2 Compactness in neutrino mass spectrum

The seesaw relation (3.6), together with the $SO(10)$ inspired constraint (3.2), leads to some difficulties in the production of an efficient leptogenesis. In fact, due to the structure of M_D, it implies a strong hierarchy in the masses for the right-handed, with the lightest right-handed (which is generally the main contributor to leptogenesis) having a mass $M \ll 10^9\ GeV$. This is well below the DI bound on the neutrino masses which are efficient for leptogenesis. Various ways have been found to get rid of this problem; the two main proposals are that the two heavier neutrinos contribute enough to produce by themselves an efficient leptogenesis or that the masses of the right-handed neutrinos can be constrained to be so close to one another to give rise to a resonant mechanism. The road which has been explored in our work has been somewhere in between; we have imposed a fine tuning on the parameters of the mass matrices, leading to a mass spectrum which, though not degenerate, is compact in form with all neutrino masses ranging around $10^9 - 10^{13}\ GeV$.

To derive the conditions necessary to obtain such a compact spectrum we first notice that (3.6) can be rewritten in the form:

$$M_R = -V_L^\dagger M_D^{diag} A M_D^{diag} V_L^* \tag{3.10}$$

where:

$$A = V_L^* M_\nu V_L^\dagger \tag{3.11}$$

For simplicity, let us approximate here V_{CKM} with the identity matrix; in fact, since we are only interested in producing a spectrum which is not hierarchical in its orders of magnitude, this approximation will not significantly affect the results of our analysis.

The right-handed neutrino mass matrix then becomes:

$$M_{Rik} \sim -A_{ik} M_{Di} M_{Dk} \tag{3.12}$$

By the postulated quark-lepton symmetry, the Dirac masses satisfy the inequality $M_{D3} \gg M_{D2} \gg M_{D1}$, and the approximate equality $M_{D3} M_{D2} \sim M_{D2}^2$. A compact spectrum can then be obtained if the matrix A is such that:

$$\left|\frac{A_{33}}{A_{22}}\right| \leq \frac{M_{D2}^2}{M_{D3}^2}, \quad \left|\frac{A_{23}}{A_{22}}\right| \leq \frac{M_{D2}}{M_{D3}} \tag{3.13}$$

In this way, the entries of the matrix which would be hierarchically large due to largeness of M_{D3} are suppressed.[2] In the following work, we have assumed even more stringent conditions by requiring:

$$A_{23} = A_{33} = 0 \tag{3.14}$$

By explicit computation one can now verify that, to first order in the small quantity $\frac{M_{D1}^2}{M_{D3}^2}$, two of the three eigenvalues of the right-handed neutrino mass matrix are degenerate.

We stress that the choice of setting the two matrix elements above equal to zero is a simplicity choice, and, in general, they are only required to satisfy the weaker condition (3.13).

We can now relax the condition that the CKM matrix coincides with the identity, since, as we said before, we do not expect this choice to change the order of magnitude of the M_R eigenvalues, its elements being of order 1.

The condition (3.14) on the matrix A translates into a condition which must be satisfied by the physical parameters of M_ν. We recall that this matrix is completely specified by 3 neutrino masses, 3 mixing angles, 1 Dirac phase and 2 Majorana phases; the two conditions above give 4 equations for the complex matrix elements of the matrix A, and allow us to express 4 of the above parameters in terms of the other. The obvious choice is to find the two Majorana phases α and β (which would be otherwise unobservable), the CP-violating phase δ and the mass of the lightest neutrino m_1 (which cannot be determined through mixing experiments).

The analytical expression for the elements of A of interest becomes slightly simplified under the hypothesis that V_{CKM} is parameterized through a single non vanishing mixing angle; this is a good approximation, since we know that one of the mixing angles is much larger than the other. In this approximation:

$$V_{CKM} = \begin{pmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.15}$$

Equations (3.14) then read:

$$\frac{m_2}{m_1} e^{-2i\alpha} = -\frac{\left(c_{12}s_{23} + e^{i\delta} s_{12}s_{13}c_{23}\right)\left[s_c c_{12}s_{13}s_{23} + e^{i\delta}\left(s_c s_{12}c_{23} - c_c c_{12}c_{13}\right)\right]}{\left(-s_{12}s_{23} + e^{i\delta}c_{12}s_{13}c_{23}\right)\left[-s_c s_{12}s_{13}s_{23} + e^{i\delta}\left(c_c s_{12}c_{13} + s_c c_{12}c_{23}\right)\right]} \tag{3.16}$$

$$\frac{m_3}{m_1} e^{-2i\beta} = \frac{c_{13}c_{23}\left[s_c s_{12}c_{23} + c_{12}\left(-c_c c_{13} + s_c e^{-i\delta} s_{13}s_{23}\right)\right]}{\left(-s_{12}s_{23} + e^{i\delta}c_{12}s_{13}c_{23}\right)\left(s_c c_{13}s_{23} + c_c e^{i\delta} s_{13}\right)} \tag{3.17}$$

with the obvious notation $c_x = \cos\theta_x$, $s_x = \sin\theta_x$.

[2] This result has been derived by straightforward calculation which is analyzed in more detail in [44].

3.3 Numerical results: compactness of the spectrum

A first part of our work has consisted in the analysis of the constraints imposed by (3.14) on the parameters of the PMNS matrix. In previous works [44] the analysis was done in the hypothesis that the values of the mixing angles were set to their best fit values. In order to allow for more freedom, in this work we have instead randomly generated values of these angles inside the experimentally allowed range of values centered around the best fit values with a width of 3σ. The choice of the signs of the angle deserves some comments; from the form of the PMNS matrix it can be seen that a change in sign of θ_{13} can be absorbed into a change $\delta \to \delta + \pi$, since they always appear in the final matrix in the combination $\sin\theta_{13}e^{i\delta}$. The situation is slightly more complicated for the other angles. The crucial observation [49] lies in the fact that both Dirac and Majorana spinors can be subject to the transformation $\psi \to -\psi$ without changing the physical content of the theory; then it can easily be proven that the change $\theta_{12} \to -\theta_{12}$ and $\theta_{23} \to -\theta_{23}$ merely amounts to a redefinition of the signs of the spinors. For this reason, only positive values of the angles have been generated.

The values of the squared mass differences were known from experiments on mixing; therefore, for each set of PMNS mixing angles, one can solve the compactness conditions to deduce the mass of the lightest neutrino m_1, the CP violating phase δ and the Majorana phases α and β. We anticipate that some of the results in this section will be superseded by the stronger leptogenesis constraint which will be imposed in Section 3.5. Such constraint, however, is to be taken *cum granu salis*; due to our decision of using $SO(10)$ just as an inspiration for the mass rules which we adoperated, we have to take into account the possibility that a consistent treatment of leptogenesis in a real $SO(10)$ model may modify the results of Section 3.5 (for a further discussion of this point we refer the reader to the Conclusions). On the other hand, the results of this Section, being based only on the mass rules of $SO(10)$ and on the assumption that charged leptons mass eigenstates coincide with weak eigenstates, are expected to change less; for this reason, we have decided to report these results even though many of them will be later modified by the further constraint of leptogenesis.

By taking the absolute value of (3.16) and (3.17) we find two coupled equations which do not contain the Majorana phases. For a given set of mixing angles, these two conditions translate into two curves in the $m_1 - \delta$ plane, of which we need to find the intersections; a typical case, examined in [44], is shown in Figure 3.1, where the values of δ are restricted to a range of $[-\pi, \pi]$. As can be seen, two solutions are admitted with nearly opposite values of δ and the same values for m_1.

For each of these two solutions, by taking the imaginary parts of (3.14) we can solve for the Majorana phases, which are also nearly opposite in the two cases; the same work has been repeated for each of the generated values of the

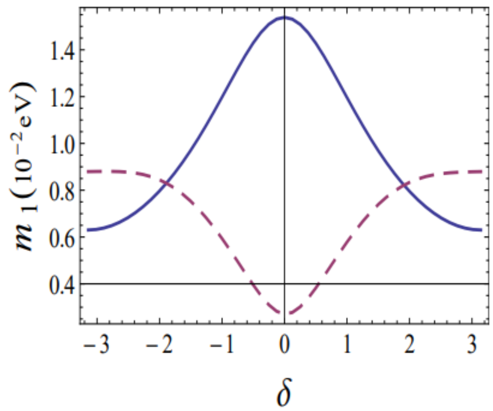


Figure 3.1: Graphical method of solution of the compactness conditions in the $m_1 - \delta$ plane (adapted from [44]).

mixing angles.

After the generation, we have looked for correlations between admitted values for the parameters by plotting the points against various choices of the PMNS parameters. In Figure 3.2 we have represented in the plane $m_1 - \delta$ the solutions which have been found for all the randomly generated values of the mixing angles.

Since we were particularly interested in constraints on the mixing angles, in Figure 3.3 we represent the points corresponding to the best fit value of δ (of course they are taken actually from a very thin strip centered around the best fit value of δ of width 0.01) in the plane of the angles θ_{12} and θ_{23}. The reason for this particular choice is that the points here showed, as is seen from the figure, a linear correlation.

The linear correlation suggested by Figure 3.3, however, is evidenced in a region which falls outside the confidence region for the mixing angles. Since these mixing angles are determined with much better accuracy than the Dirac CP violating phase, it is of interest to try to give a prediction for δ based on the requirement that the points enter the confidence region; it is found that this happens for $1.78\pi \leq \delta \leq 1.80\pi$ (compare with the best fit value 1.37π). The results are shown in Figure 3.4. Constraining the points to lie in the rectangle of intersection of the two confidence regions for both θ_{12} and θ_{23} induces a further inequality on m_1, which is bound to lie between 0.0021 eV and 0.0026 eV.

3.4 Numerical solutions of Boltzmann equations

For each parameter set leading to a compact spectrum the Boltzmann equations deduced in Chapter 2 have been solved; it is the purpose of this section to

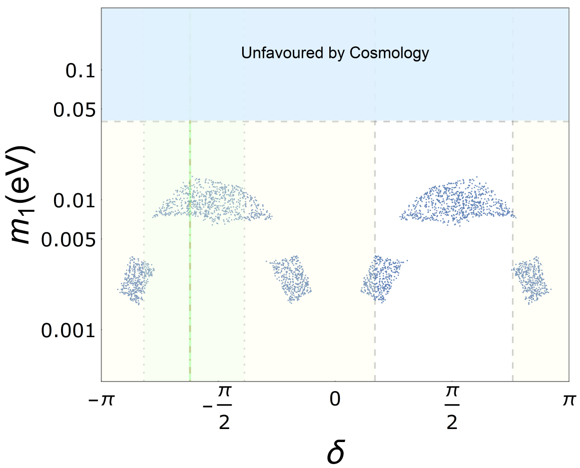


Figure 3.2: Generated points corresponding to compact neutrino mass spectrum; the green region corresponds to the 1σ confidence region, the yellow region corresponds to the 3σ confidence region and the blue dashed line is the best fit value for δ. The cosmological constraints which rule out the upper region descends from the requirement that neutrinos do not close the Universe ($\Omega_\nu < 1$).

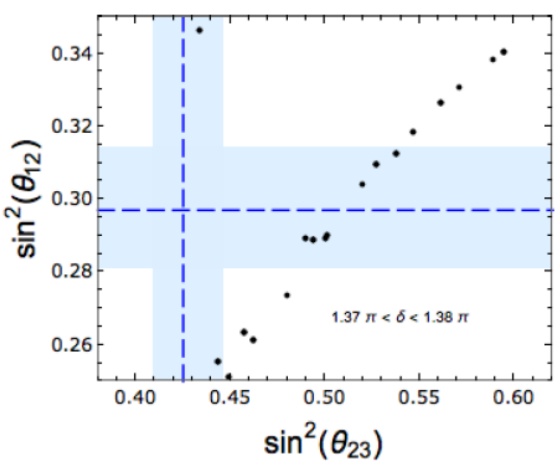


Figure 3.3: Generated points corresponding to compact neutrino mass spectrum and to the best fit value of δ; the blue regions represent the 1σ confidence regions.

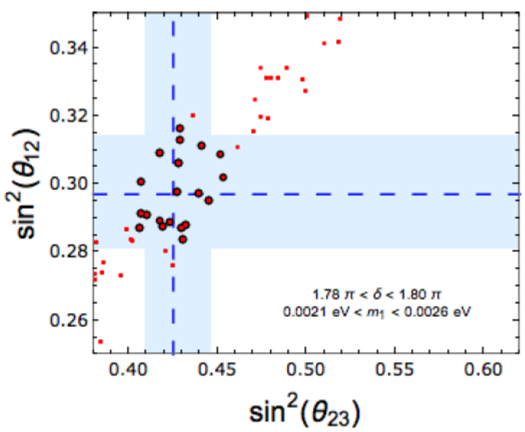


Figure 3.4: Generated points corresponding to compact neutrino mass spectrum and $1.78\pi \leq \delta \leq 1.80\pi$.

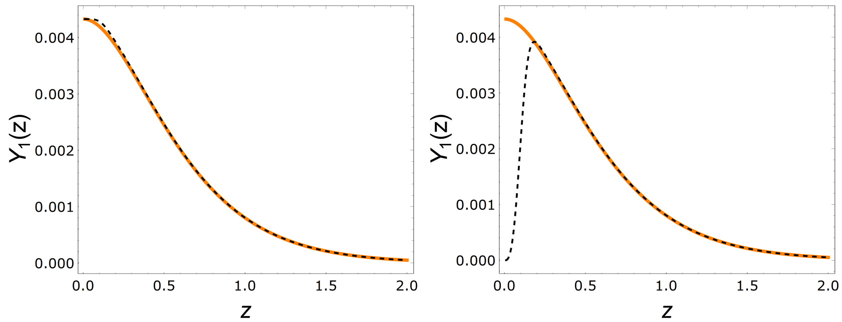
describe the numerical work involved in such a task.

The differential equations for the right-handed neutrino concentrations are reproduced here for convenience:

$$sHz\frac{dY_i}{dz} = -\gamma_i\left(\frac{Y_i}{Y_{ieq}} - 1\right) \tag{3.18}$$

where we recall that $z = \frac{M}{T}$, M being a convenient mass scale; in our cases, the neutrino mass spectrum being degenerate, we have chosen M to be 10^9 GeV, the typical order of magnitude for right-handed neutrinos. With this choice leptogenesis is expected to happen at $z \sim 1$, when right-handed neutrinos become non relativistic and decouple.

(3.18) form a set of three independent linear differential equations with non constant coefficients with the same form of the Boltzmann equations involved, for example, in the freeze-out of a cold relic [50]. They admit the same qualitative behavior, with the solutions reaching asymptotically Y_{ieq}. In Figure 3.5 we represent the behavior of $Y_i(z)$ for two different initial conditions to prove the independence of this result; in both cases the solution approaches, in a time which is very short relative to the timescale of leptogenesis ($z \sim 1$), the equilibrium solution. It is not a trivial consequence that the final result for the leptonic yield should be independent of the initial conditions on the right-handed neutrinos; we have verified such independence explicitly by using both initial conditions to compute the yield. The equations have been numerically solved using the software Mathematica©.



(a) (b)

Figure 3.5: Numerical evolution compared with equilibrium distribution for (a) thermal initial conditions and (b) vanishing initial conditions; the equilibrium distribution is shown in orange, the thermal distribution is shown in black.

We now reproduce the equations for $B - L$ asymmetries:

$$sHz\frac{dY_{\Delta\alpha}}{dz} = -\sum_i \epsilon_{i\alpha}\gamma_i\left(\frac{Y_i}{Y_{ieq}} - 1\right) + \sum_i \frac{\gamma_{i\alpha}}{2}\left(\frac{\mathcal{L}_\alpha}{Y_{\alpha eq}} + \frac{\Phi}{Y_{\phi eq}}\right) \tag{3.19}$$

These are a set of coupled linear differential equations; both the coefficients of the homogeneous part and the non homogeneous terms vanish asymptotically, which implies the possibility of a non vanishing asymptotic yield. The non homogeneous terms depend on the solution of (3.18), so they are known numerically.

In the numerical treatment of this system a number of difficulties have been encountered; they are all linked with the so called *stiffness* problem of the system [51]. To explicitate the existence of this problem we now turn to an explicit decoupling of the equations. We can rewrite the system, with obvious notation, as:

$$\frac{dY_{\Delta\alpha}}{dz} = M_{\alpha\beta}(z)Y_{\Delta\beta}(z) + f_\alpha(z) \tag{3.20}$$

The matrix $M_{\alpha\beta}$ is not symmetrical; it has, however, real eigenvalues $\lambda_a(z)$ and eigenvectors $v_\alpha^{(a)}(z)$. The dependence of the eigenvectors on z is actually so slight as to be undetectable by numerical work because of the compactness in the mass spectrum; it can be easily seen, by explicitating the form of $M_{\alpha\beta}$, that this simplification derives from the compactness of the spectrum, which makes the three equilibrium distributions for the species nearly equal. In the following, we will consider $v_\alpha^{(a)}$ to be independent of z. We can decompose the solution over these eigenvectors:

$$Y_{\Delta\alpha}(z) = \sum_a c_a(z)v_\alpha^{(a)}(z) \tag{3.21}$$

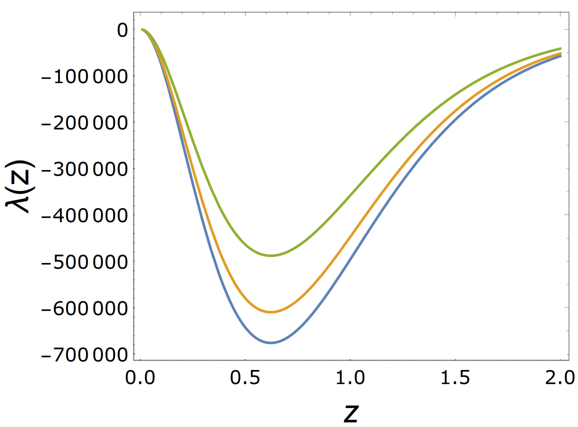


Figure 3.6: Numerical prediction for the three eigenvalues $\lambda(z)$ for a typical case.

We can use the set of vectors:

$$\mathbf{u}^a = \epsilon^{abc}\mathbf{v}^{(b)} \times \mathbf{v}^{(c)} \tag{3.22}$$

to project the equations for $Y_{\Delta\alpha}$ into three decoupled equations for the coefficients of the decomposition of the form:

$$\frac{dc_a}{dz} = \lambda_a(z)c_a(z) + \phi_a(z) \tag{3.23}$$

where we have used $M_{\alpha\beta}v^{(a)}_\beta = \lambda_a v^{(a)}_\alpha$ and, by straightforward substitution, one finds:

$$\phi_a(z) = \frac{f_\alpha(z)u^a_\alpha}{v^{(a)}_\alpha u^a_\alpha} \tag{3.24}$$

The behavior of $\lambda_a(z)$ is represented in Figure 3.6 for a typical case; the negative sign denotes that the solution will settle to an equilibrium. However, the absolute value of the maximum reaches values of order $\sim 10^6$; a consequence is that the timescale over which the solution undergoes consistent changes is $\delta z \sim 10^{-6}$. A numerical algorithm using a step larger than this quantity will necessarily produce solutions which do not reproduce the true behavior and in general lead to fictitious divergences. The fact that we are interested in much larger timescale ($z \sim 1$) justifies the stiffness problems encountered by the numerical algorithm proposed by Mathematica©; in fact, using s step $\delta z \sim 10^{-6}$ over such timescales requires a running time of days for a single data set. We have therefore decided to follow a different path to the solution of the system.

It is well known that a linear system of the form (3.23) admits an exact solution in the form:

$$c_a(z) = e^{\int_{z_{min}}^{z} \lambda_a(z')dz'} \left(c_a(z_{min}) + \int_{z_{min}}^{z} \phi_a(z') e^{-\int_{z_{min}}^{z'} \lambda_a(z'')dz''} dz' \right) \quad (3.25)$$

Some observations are in order here. First we notice that the integral of λ_a over z from z_{min} to $+\infty$ converges; were it not so, the exponential factor before the parenthesis would make the asymptotic yield vanish (recall that λ_a is always negative). At the same time, due to the magnitude of the maximum of $|\lambda_a|$, the value of this integral is extremely large, which causes the contribution of the initial condition to be unimportant in determining the asymptotic value of the yield. The determination of the solution through the use of (3.25) is still not trivial from a numerical standpoint; in fact, the magnitude of the exponents, for z far enough from z_{min}, would cause overflow problems in the computation of the integrals. Such problems are interpreted by Mathematica© as fictitious divergences of the integrals. We can explicitly cancel these divergences by writing (3.25) in the equivalent form[3]:

$$c_a(z) = \int_{z_{min}}^{z} \phi_a(z') e^{\int_{z'}^{z} \lambda_a(z'')dz''} dz' \quad (3.26)$$

This form of the solution contains only integrals which, for z far enough from z_{min}, can be done numerically without overflow problems; notice, however, that they are affected by the same trouble for z near z_{min}, where the previous form was more useful. It follows that the complete solution requires a separation of the interval in z into a region near z_{min} and a region far from it; an example of a sample of the points of our solution for a typical case is given in Figure 3.7. Since we are interested only in the asymptotic value, however, we can directly use (3.26) to obtain it (for $z \to +\infty$) without any need for the intermediate evolution (which is the main advantage of this treatment).

To increase confidence in our method, we have also computed the value of the integral approximately; the form of the integrand, in fact, makes it an ideal field of application of the saddle point approximation. It has been found that the two results, the one obtained by numerical evaluation of the integral and the one obtained by saddle point approximation, differ to within an error of 10%.

3.5 Numerical results: leptogenesis

We now describe the results obtained from our numerical method. Only a small number of the set of parameters leading to a compact spectrum is able to guarantee an efficient leptogenesis; in fact, only 54 out of 1661 points produced a baryonic yield which is consistent with the theoretical value in a 3σ confidence range.

In the same way as we did for the compact neutrino masses data, we plotted these 54 points against various sets of parameters, in order to look for correlations. The linear behavior which had been found in Figure 3.3 for the variables

[3]The contribution of the initial conditions will not be written anymore

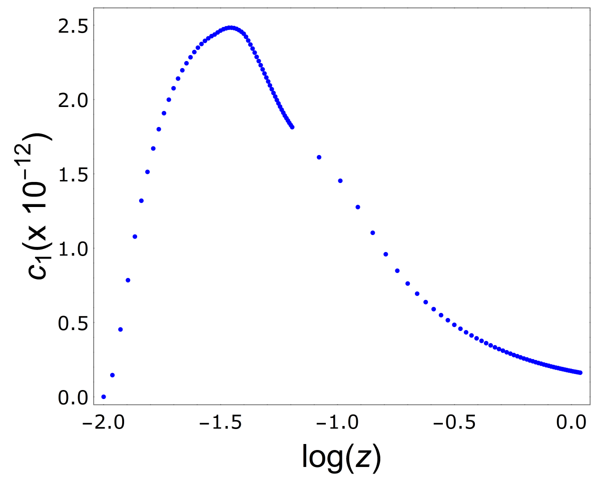


Figure 3.7: Sample of the numerical solution of $c_1(z)$ for a typical case.

$\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ *for a fixed value of* δ is not observed now; however, we have represented in Figure 3.8 the points on the same axes *for all values of* δ. It can be seen that an efficient leptogenesis is not obtained in the lower right quadrant of the plot (the dashed lines are the best fit values of the parameters). A physical conclusion is that if $\sin^2\theta_{12} \leq 0.3$ then $\sin^2\theta_{23}$ is bound to be $\sin^2\theta_{23} \leq 0.5$.

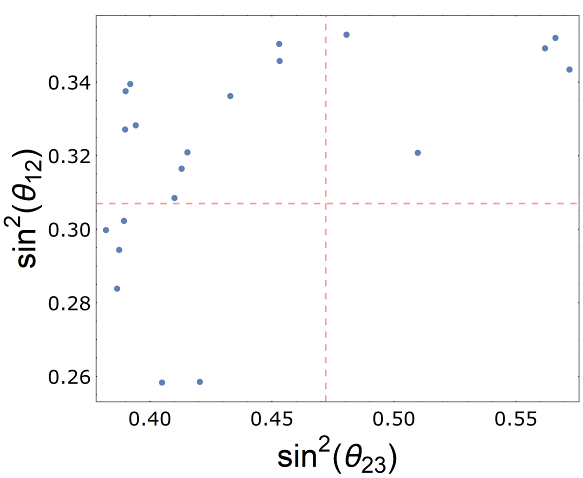


Figure 3.8: Points corresponding to a baryonic yield in a range of 3σ from the experimental value in the plane $\sin^2\theta_{12} - \sin^2\theta_{23}$.

An interesting correlation has been found in the $\alpha - \beta$ plane for the points leading to a compact mass spectrum; these are represented in Figure 3.9. It is found that they lie only in two narrow regions of the parameter α; in particular, the best fit value of α can be estimated to be $|\alpha| \sim \frac{\pi}{2}$, while β is found to lie only in narrow regions centered around the values $\beta \sim 0.3\pi$ and $\beta \sim -0.7\pi$.

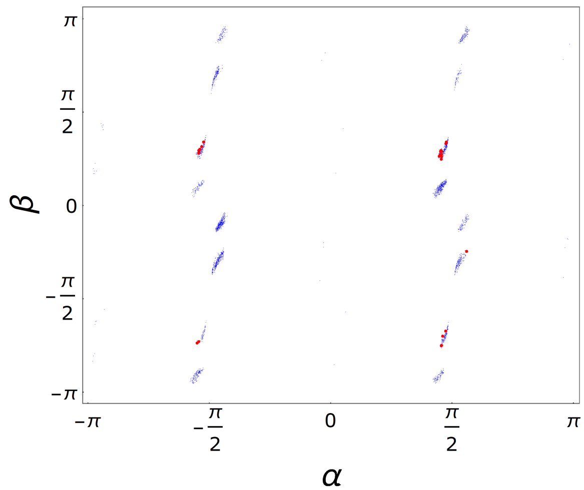


Figure 3.9: Numerical results in the $\alpha-\beta$ plane; the points leading to a compact mass spectrum are shown in blue, the points leading to a baryonic yield in a 3σ range from the experimental value are shown in red.

The plot given in Figure 3.2 has been modified to include the constraint coming from leptogenesis; the result is given in Figure 3.10. Of the regions which were initially populated by points with a compact mass spectrum, only the lower ones survive the leptogenesis constraint.

For this reason, the possibility of giving a confidence region for the parameter m_1 on the basis of purely leptogenesis constraint is admitted; we represent in Figure 3.11 the points in the $m_1 - Y$ plane. It is evident that all the points leading to an efficient leptogenesis have an m_1 between 0.0017 eV and 0.0035 eV. From the correlation between m_1 and δ analyzed in Figure 3.10 it is also possible to find regions in which δ lies, which are centered around the values $\delta \sim -0.79\pi$ and $\delta \sim 0.23\pi$[4].

We looked for similar exclusion region for the values of the mixing angles θ_{12}, θ_{13} and θ_{23}; however, we did not find any, and therefore we conclude that we are not able to give informations, through our leptogenesis model, on these angles separately, but only through the correlations analyzed above.

A final plot, reported in Figure 3.12, represents the Jarlskog invariant [52], defined as $J = \mathrm{Im}\{U_{12}U_{23}U_{13}^*U_{32}^*\}$, in the J_Y plane; the advantage of the use of this quantity is its independence on the parameterization of the PMNS matrix. From the plot we can deduce that the Jarlskog parameters which can give an

[4]The corresponding regions with $\delta \to -\delta$ are admitted as well.

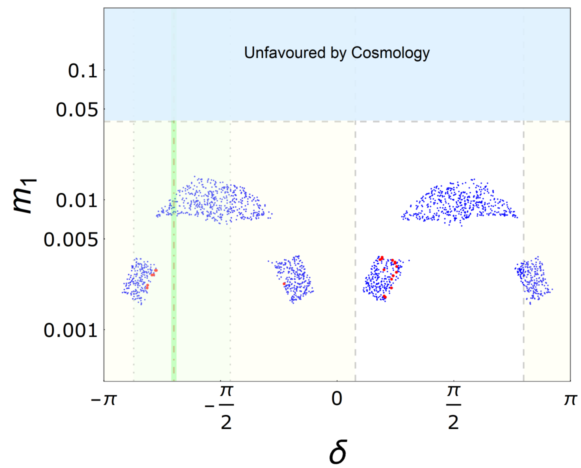


Figure 3.10: Numerical results in the $m_1 - \delta$ plane; the points leading to a compact mass spectrum are shown in blue, the points leading to a baryonic yield in a 3σ range from the experimental value are shown in red.

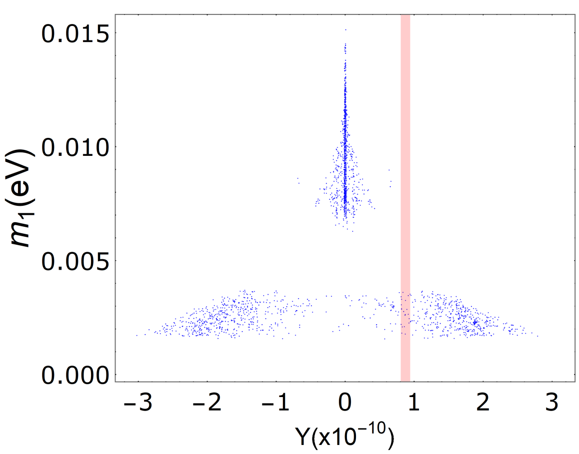


Figure 3.11: Numerical results in the $m_1 - Y$ plane; the points leading to a compact mass spectrum are shown in blue while the 3σ confidence region is shown in red.

efficient leptogenesis are in the region $0.017 \leq |J| \leq 0.024$. The sign of J is of course extremely important since it determines the sign of Y; this explains the particular symmetry structure of the plot.

We finally add that there is an extremely narrow region, of which we have managed to find a single point, that does not intersect the previously stated intervals; this can be qualitatively characterized by the fact that the Jarlskog invariant is negative, in contrast to the other solutions.

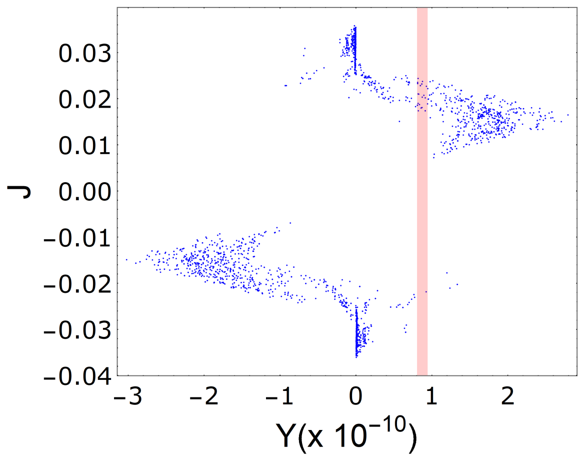


Figure 3.12: Numerical results in the $J - Y$ plane; the points leading to a compact mass spectrum are shown in blue while the 3σ confidence region is shown in red.

Conclusions

Leptogenesis is an attractive model which is able to handle the problem of baryon asymmetry in the Universe using the same seesaw mechanism proposed to describe the neutrino masses. The strongest obstacle to its predictive power is the insensitivity of low energy physics to the seesaw parameters; this prevents us from having information on the Dirac neutrino masses and, therefore, from giving an estimate of the BAU predicted from the model. Only qualitative expectations, such as the hierarchy in the Dirac mass matrix, due to the similarity with the charged leptons mass matrix, are available at the present.

A way out of this *hiatus* is to try to estimate the seesaw parameters using mass relations coming from higher symmetry groups; the main example given in this thesis was a specific mass relation obtained in an $SO(10)$ inspired context. After this subsidiary information is added to the model, the way is open to predicting explicitly the baryonic yield. A difficulty comes from the natural hierarchy which would easily appear in the right-handed neutrino spectrum, which mirrors that appearing in the Dirac mass matrix; such a hierarchy brings the mass of the lightest neutrino, which is responsible for leptogenesis, below the Davidson-Ibarra limit. Different solutions have been proposed for this problem; the one analyzed in this work makes recourse to the imposition of a fine tuning on the left-handed neutrino mass parameters leading to a compact mass spectrum. Once this difficulty has been overcome, the baryonic yield can be predicted and compared to the experimental value to obtain bounds on the mass parameters.

In this work we have concentrated on proving the possibility of deriving constraints on low energy physics parameters from the requirement of consistent leptogenesis. We found that it was possible to restrict the range of some of the parameters of the PMNS matrix; in particular, m_1 was found to lie between $0.0017\ eV$ and $0.0035\ eV$; δ lies in regions centered on the values of $\delta \sim -0.79\pi$ and $\delta \sim 0.23\pi$; the Jarlskog invariant lies in the interval $0.017 \leq |J| \leq 0.024$; the Majorana phase α lies around the value of $|\alpha| \sim \frac{\pi}{2}$, while β is found in narrow regions around the values $\beta \sim 0.3\pi$ and $\beta \sim -0.7\pi$. A final result suggested by our analysis is the absence of population of one of the octants in the $\sin^2\theta_{12} - \sin^2\theta_{23}$ plane.

Some intermediate results were also presented which do not make use of the leptogenesis constraints but only requires the compactness of the right-handed neutrino mass spectrum. The reason why these results may be considered of interest is that they depend only on the $SO(10)$ inspired assumption, and it is likely that they will not be modified in a further analysis which treats in a detailed way all of the consequences of a real $SO(10)$ model. A specific pattern was found in $m_1 - \delta$ plane; moreover, a linear correlation was determined between the variable $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ for fixed values of the Dirac phase δ.

We conclude by underlining that the aim of the present work was not as much of practical importance in determining confidence ranges of parameters, the hypothesis of mass relations being not completely justified; we were instead interested in proposing an approach to leptogenesis which is able, even in the context of hierarchical Dirac masses in which leptogenesis seemed to fail, to connect the low energy neutrino mass physics to the high energy physics identified by the BAU and the seesaw parameters.

Appendix A

CP asymmetries and decay rates

A.1 Feynman rules for Majorana fermions

A Majorana fermion may be regarded as a fermion possessing the same free Lagrangian as a Dirac spinor:

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{A.1}$$

with the constraint:

$$\psi = \mathcal{C}\psi^* \tag{A.2}$$

The operator $\mathcal{C}$ is a charge conjugation operator which depends on the representation and is required to satisfy:

$$\mathcal{C}\gamma^{\mu *} = -\gamma^\mu \mathcal{C} \tag{A.3}$$

In the Dirac representation γ^2 is the only matrix with imaginary elements, so that we can take $\mathcal{C} = \gamma^2$ (a great deal of literature is devoted to proving the invariance of Feynman rules under different representations; we will use this one without worrying about such problems).

The quantization of the field can now proceed by the usual methods through an expansion of the field operator into solutions of the free Majorana lagrangian; it is easy to see that the constraint (A.2) requires the field to take the form:

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_\lambda \left[\xi_{\mathbf{p}\lambda} a_{\mathbf{p}\lambda} e^{-ip\cdot x} + \gamma^2 \xi^*_{\mathbf{p}\lambda} a^\dagger_{\mathbf{p}\lambda} e^{ip\cdot x}\right] \tag{A.4}$$

where $\xi_{\mathbf{p}\lambda}$ are the two solutions, indicized by $\lambda = 1{,}2$, of the equation $p\!\!\!/\xi_{\mathbf{p}\lambda} = m\xi_{\mathbf{p}\lambda}$ and on-shell conditions are implicitly supposed for the four-vector p.

Deduction of the rules now proceeds as usual, by expanding the S operator in terms of the interaction vertices; the main difference between the Dirac case is that now the same species can be created both by the operator $\psi(x)$ and $\overline{\psi}(x)$. This means that the Wick theorem now translates into the statement that every matrix element can be reduced to the calculation of matrix elements of the following operators: $\langle 0| T \left\{\overline{\psi}(x)\psi(0)\right\}|0\rangle$,$\langle 0| T \left\{\psi(x)\psi(0)\right\}|0\rangle$,$\langle 0| T \left\{\overline{\psi}(x)\overline{\psi}(0)\right\}|0\rangle$. We can now use the above expansion and the usual commutation rules to write down an expression for the four-dimensional Fourier transforms of these functions; these turn out to be proportional to sums of the form $\sum_\lambda \xi_{\mathbf{p}\lambda}\overline{\xi}_{\mathbf{p}\lambda}$ and $\sum_\lambda \xi^T_{\mathbf{p}\lambda}\gamma^2\gamma^0\overline{\xi}_{\mathbf{p}\lambda}$. These sums can be explicitly calculated, for example, in the case that the three-dimensional impulse is directed along the z axis; the result can then be extended to a relativistic invariant form. The calculations lead to the following results:

$$
\begin{aligned}
\langle 0| T \left\{\overline{\psi}(x)\psi(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\left(\not{p}+m\right)e^{-ip\cdot x} \\
\langle 0| T \left\{\psi(x)\psi(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\gamma^2\gamma^0\left(\not{p}-m\right)e^{-ip\cdot x} \\
\langle 0| T \left\{\overline{\psi}(x)\overline{\psi}(0)\right\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\left(\not{p}-m\right)\gamma^2\gamma^0 e^{-ip\cdot x}
\end{aligned}
\tag{A.5}
$$

These results could also have been obtained through a path integral formulation; we give a sketch of the derivation.

The generating functional of the Green functions is obtained by adding to the Lagrangian a term of the form $\overline{j}(x)\psi(x)+h.c.$; the constraint that the spinor is of Majorana type can be imposed by inserting into the integral a representation of a δ function:

$$
\begin{aligned}
\mathcal{Z}\left[j,\overline{j}\right] = e^{-iW\left[j,\overline{j}\right]} &= \int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS\left[\psi,\overline{\psi},j,\overline{j}\right]}\prod_x \delta\left(\psi(x)-\gamma^2\psi^*(x)\right) \\
&= \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}\phi e^{iS\left[\psi,\overline{\psi},j,\overline{j}\right]+i\int d^4x\phi(x)\left(\psi(x)-\gamma^2\psi^*(x)\right)}
\end{aligned}
\tag{A.6}
$$

where of course the action is:

$$
S\left[\psi,\overline{\psi},j,\overline{j}\right] = \int d^4x\left[\overline{\psi}\left(i\not{\partial}-m\right)\psi+\overline{j}\psi+\overline{\psi}j\right] \tag{A.7}
$$

The auxiliary field ϕ has been introduced as a spinor field with no conjugate field. The integrals over ψ and $\overline{\psi}$ can be performed in the usual way to obtain a functional of ϕ, j and $\overline{j}$; the subsequent integral over ϕ can be easily performed since the integrand is still the exponential of a quadratic functional of ϕ. The calculation are lengthy but straightforward and they will not be done here, since they lead to the same result of the second quantization treatment for the Green functions.

A.2 Calculation of leptogenesis amplitudes

We will first compute the total decay rate (into leptons and antileptons) of the right-handed neutrino through the interaction Lagrangian:

$$\mathcal{L}_I = -\lambda_{\alpha i}\bar{l}_\alpha N_i \phi^c - h.c. \tag{A.8}$$

At tree level the amplitude for decay into l_α and $\overline{\phi}$ (Figure 2.1(a)) is:

$$-i\lambda_{\alpha i}\overline{u}_\alpha \xi_i \tag{A.9}$$

where u_α and ξ_i are respectively the free final and initial spinors of the lepton and the neutrino respectively. By squaring, summing over lepton polarizations and averaging over neutrino polarizations, we easily find the result:

$$\Gamma_{N_i \to l_\alpha + \phi} = \frac{M_i |\lambda_{\alpha i}|^2}{16\pi} \tag{A.10}$$

After multiplying by 2 to take into account the decay both in leptons and antileptons, we find the result stated in the text for Γ_i.

Let us now turn to the calculation of the CP asymmetry. The diagrams should contain contributions from all neutrino species; however, the contribution to the ith vertex and wavefunction coming from an intermediate state containing the ith neutrino itself contains no CP asymmetry, since it will contain a factor of $\mathrm{Im}\Big\{|\lambda_{\alpha i}|^2 |\lambda_{\beta i}|^2\Big\}$.

Taking into account the loop diagrams, we can write the squared matrix elements for the decays in the following form:

$$\begin{aligned} |\mathcal{M}_{N_i \to l_\alpha + \phi}|^2 &= |c_0 T_0 + c_1 T_1|^2 = |c_0 T_0|^2 + 2\,\mathrm{Re}\{c_0^* T_0^* c_1 T_1\} \\ \left|\mathcal{M}_{N_i \to \bar{l}_\alpha + \overline{\phi}}\right|^2 &= |c_0^* T_0 + c_1^* T_1|^2 = |c_0 T_0|^2 + 2\,\mathrm{Re}\{c_0^* T_0 c_1 T_1^*\} \end{aligned} \tag{A.11}$$

where c_0 and c_1 are the coupling constant part of the amplitude and T_0 is the kinematic part. The difference between these two matrix elements will then be $-4\,\mathrm{Im}\{c_0^* c_1\}\,\mathrm{Im}\{T_0^* T_1\}$. Let us evaluate this quantity for the three diagrams in Figure 2.1(b); we reproduce them in Figure A.1 for convenience with the kinematic variables used in the calculations.

A.2.1 Vertex correction

The first diagram in Figure 2.1(b) represents a correction to the interaction vertex whose amplitude takes the form:

$$\sum_{\beta, j\neq i} \lambda_{\alpha j}\lambda_{\beta j}^* \lambda_{\beta i} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - \not{q} - M_j} \frac{1}{\not{k}} \frac{1}{(P-k)^2} \tag{A.12}$$

(the $i\epsilon$ are not explicitly shown). The tree amplitude is $-i\lambda_{\alpha i}$, so we are interested in the real part of the integral in (A.12); these are obtained ([53]) by

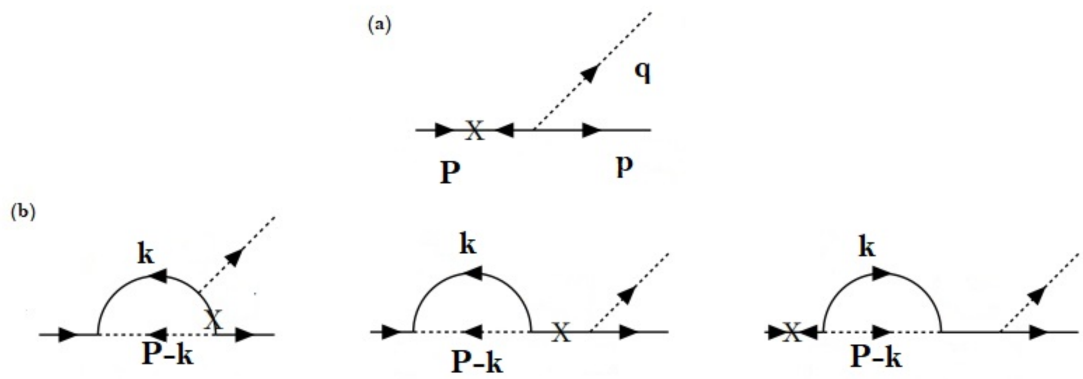


Figure A.1: Diagrams contributing to order $\mathcal{O}(\lambda^2)$ in the amplitude to right-handed neutrino decay (a) at tree level and (b) at one loop level with the kinematic variables indicated.

substituting two of the three propagators by δ functions which keep the particles on shell. It is easy to see that the right-handed neutrino cannot be on shell together with another one of the particles at the same time; therefore the difference between the matrix elements will be:

$$-4\sum_{\beta,j\neq i}\mathrm{Im}\{\lambda_{\alpha i}\lambda^*_{\alpha j}\lambda_{\beta j}\lambda^*_{\beta i}\}\pi^2\int\frac{d^4k}{(2\pi^4)}\frac{1}{\not{k}-\not{q}-M_j}\not{k}\delta\left(k^2\right)\delta\left((P-k)^2\right) \tag{A.13}$$

The integrals can be done in the easiest way in the COM frame where the initial neutrino is at rest; we will not report the partial result since it will have to be added to the other diagrams corrections.

A.2.2 Wavefunction correction

The remaining diagrams represent the corrections to the external leg; in particular, the second one is characteristic of Majorana spinors, which can decay both into $l_\beta\phi$ and $\overline{l_\beta\phi}$. The corresponding amplitudes are:

$$\begin{gathered}\sum_{\beta,j\neq i}\lambda^*_{\beta i}\lambda_{\beta j}\lambda^*_{\alpha j}\int\frac{d^4k}{(2\pi)^4}\frac{1}{\not{P}-M_j}\frac{1}{\not{k}}\frac{1}{(P-k)^2}\\ \sum_{\beta,j\neq i}\lambda_{\beta i}\lambda^*_{\beta j}\lambda^*_{\alpha j}\int\frac{d^4k}{(2\pi)^4}\gamma^2\gamma^0\frac{1}{\not{P}+M_j}\frac{1}{\not{k}}\frac{1}{(P-k)^2}\end{gathered} \tag{A.14}$$

The real part of the amplitudes is obtained by setting both the lepton and the Higgs on shell and substituting the two propagators with the corresponding δ. The final integrals can be done in the COM frame.

To compare the results with the formulas given in Chapter 2 it remains to integrate over the phase space; the calculations are straightforward although lengthy.

A.2.3 Degenerate masses

A final point which needs to be taken into account is the possibility of nearly degenerate masses. In this case, in fact, the wavefunction corrections are affected by a small denominator (in the limit of perfect degeneracy it apparently vanishes) which causes divergences in the above formulas. The physical cause for these divergences lies in the fact that, in the limit of perfect degeneracy, the two degenerate neutrinos can be regarded as two effective states of the same particle which are mixed by the interaction; therefore, they cannot be used as rigorous in and out states in the S matrix formulation. Instead, we should write the complete set of Green functions $\langle 0| T\left\{\overline{N}_i(x)N_j(0)\right\}|0\rangle$:

$$\begin{pmatrix} \langle 0| T\left\{\overline{N}_i(x)N_i(0)\right\}|0\rangle & \langle 0| T\left\{\overline{N}_i(x)N_j(0)\right\}|0\rangle \\ \langle 0| T\left\{\overline{N}_j(x)N_i(0)\right\}|0\rangle & \langle 0| T\left\{\overline{N}_j(x)N_j(0)\right\}|0\rangle \end{pmatrix} \tag{A.15}$$

and diagonalize it (using the Dyson equations for each propagator) to obtain the right in and out states.

We can, however, obtain the results given in Chapter 2 through a simple observation (this reasoning, however, is not to be regarded as rigorous; it is only meant to heuristically motivate the form given for the regulator. A rigorous derivation follows the path described above); the degeneracy in the masses, in fact, is lifted by the interaction, due to the different lifetimes Γ_i and Γ_j for the two different species. This enters the previous formulas in the following way. We remind the reader that, in the presence of a finite lifetime, the propagator develops an imaginary part in the pole and takes the typical Breit-Wigner form $\frac{1}{P^2-M_j^2-iM_j\Gamma_j}$. In the diagrams above this is to be substituted for the propagator of the intermediate jth right-handed neutrinos; moreover, the on shell condition $P^2 = M_i^2$ should be replaced, at a purely mathematical level, by the form $P^2 = M_i^2 + i\Gamma_i M_i$. This means that all the factors $\frac{1}{M_i^2-M_j^2}$ should be replaced by the real part of the modified propagator:

$$\frac{1}{M_i^2 - M_j^2} \to \frac{M_i^2 - M_j^2}{\left(M_i^2 - M_j^2\right)^2 + \left(\Gamma_i M_i - \Gamma_j M_j\right)^2} \tag{A.16}$$

By explicit substitution the formulas given in Chapter 2 can now be straightforwardly be derived.

Appendix B

Deduction of the Davidson-Ibarra (DI) bound

Let us start with the expressions (2.71) for the CP asymmetries in the decay of the right-handed neutrinos. Since our aim here is to prove the existence of the bound in the case of hierarchical neutrinos, we will now take the limiting form of the loop functions for the $\epsilon_{1\alpha}$ (N_1 being the lightest neutrino) for the case that $\frac{M_1}{M_{2,3}} \to 0$; the regulators in the denominator can clearly be neglected. It is easy to see that $f(\frac{M_k^2}{M_i^2}) \to -\frac{3}{2}\frac{M_1}{M_k}$, while $g(x)$ is smaller by another factor of $\frac{M_1}{M_j}$ and will be neglected. We thus obtain an estimate for the CP asymmetries:

$$\epsilon_{i\alpha} = -\frac{3M_i \sum_{k\neq j,\beta} \mathrm{Im}\left\{\lambda^\dagger_{i\alpha}\lambda_{\alpha k}\lambda^\dagger_{i\beta}\lambda_{\beta k}\right\}}{16\pi\left(\lambda^\dagger\lambda\right)_{ii} M_k} \tag{B.1}$$

Upon recognizing the seesaw formula for the mass of the left-handed neutrino mass matrix M_ν we find:

$$\epsilon_{i\alpha} = \frac{3M_i}{16\pi\left(\lambda^\dagger\lambda\right)_{ii} v^2} \sum_{k\neq i} \mathrm{Im}\left\{\lambda_{\alpha i}\left(m^*\lambda^T\right)_{\alpha i}\right\} \tag{B.2}$$

We can put an upper limit on the dot product inside the imaginary part to obtain the inequality:

$$\epsilon_{i\alpha} \leq \frac{3M_i M_{\nu max} P_{i\alpha}}{16\pi v^2} \tag{B.3}$$

This upper bound is the core result of this section; it explicitly shows that the CP asymmetry and the masses of the neutrinos are not independent. The successive step of the reasoning necessarily involves the numerical work of finding the required CP asymmetry for producing a baryon asymmetry of the same order as the observed one. This numerical work [39] implies that $\epsilon \geq 10^{-6}$. Taking for $M_{\nu max}$ the atmospheric mass, we find the bound stated in this work:

$$M_1 \geq 10^9 GeV \tag{B.4}$$

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